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Etienne Billette de Villemeur, Laurent Flochel, Bruno Versaevel

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GATE Groupe d'Analyse et de Théorie Économique
UMR 5824 du CNRS
93 chemin des Mouilles – 69130 Écully – France
B.P. 167 – 69131 Écully Cedex
Tél. +33 (0)4 72 86 60 60 – Fax +33 (0)4 72 86 60 90
Messagerie électronique gate@gate.cnrs.fr
Serveur Web : www.gate.cnrs.fr

Optimal Collusion with Limited Severity Constraint^{*}

Etienne Billette de Villemeur^{*}, Laurent Flochel^{**}, Bruno Versaevel[†]

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^{*} Corresponding author. Toulouse School of Economics, IDEI & GREMAQ, 21 allée de Brienne, 31000 Toulouse France (etienne.devillemeur@univ-tlse1.fr)

^{**} Charles River Associates International, 27 Avenue de l'opéra, 75001 Paris France (lflochel@crai.com)

[†] EMLYON Business School & CNRS, GATE, 69134 Ecully cedex France (versaevel@em-lyon.com)

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Optimal Collusion with Limited Severity Constraint

Abstract

Collusion sustainability depends on firms' aptitude to impose sufficiently severe punishments in case of deviation from the collusive rule. We characterize the ability of oligopolistic firms to implement a collusive strategy when their ability to punish deviations over one or several periods is limited by a severity constraint. It captures all situations in which either structural conditions (the form of payoff functions), institutional circumstances (a regulation), or financial considerations (profitability requirements) set a lower bound to firms' losses. The model specifications encompass the structural assumptions (A1-A3) in Abreu (1986) [*Journal of Economic Theory*, 39, 191-225]. The optimal punishment scheme is characterized, and the expression of the lowest discount factor for which collusion can be sustained is computed, that both depend on the status of the severity constraint. This extends received results from the literature to a large class of models that include a severity constraint, and uncovers the role of structural parameters that facilitate collusion by relaxing the constraint.

JEL Classification: C72; D43; L13.

Keywords: Collusion ; Oligopoly ; Penal codes.

1 Introduction

We characterize the ability of oligopolistic firms to implement a collusive strategy when their ability to punish deviations over one or several periods is limited by a severity constraint.

Firms in the same industry may increase profits by coordinating the prices they charge or the quantities they sell. In a legal context in which collusive agreements cannot be overtly enforced, and future profits are discounted, it is well known that an impatient firm may find it privately profitable to deviate for a while from a collusive strategy. This renders collusive agreements fundamentally unstable. However, firms may design non-cooperative discipline mechanisms that help implementing collusion.

Many papers investigate the structural conditions that facilitate the formation of cartels. Most theoretical contributions rely on a class of dynamic models usually referred to as supergames. These models feature a repeated market game in which firms maximize a flow of discounted individual profits by non-cooperatively choosing a price or a quantity over an infinite number of periods. When a deviation can be credibly and sufficiently “punished” via lower industry prices or larger quantities in subsequent time periods, conditions on structural parameters can be derived which, when satisfied, make collusion stable. One stream of that literature has followed Friedman (1971) by considering trigger strategies (commonly referred to as “grim” strategies), which call for reversion to the one-shot stage game Nash equilibrium forever when a deviation is detected in a previous period. A general result is that collusion can be sustained if the discount factor is above a threshold value. When firms cannot observe their competitors’ output levels, unobserved random shocks in demand can induce price wars to appear in equilibrium (see Porter (1983), Green and Porter (1984)). When all parameters and individual strategies are observable, models with various specific functional forms indicate that tacitly collusive agreements are more easily sustained with quantity-setting firms than with in price-setting oligopolists, and with highly differentiated products, for any number of firms (e.g., Deneckere (1983, 1984), Majerus (1988), Chang (1991), Ross (1992), Häckner (1994)).

A weakness common to all models of collusion with trigger strategies is that they rule out the possibility of modulating the level of punishments. More precisely, by assuming that when a deviation is detected firms revert to the Nash equilibrium of the one-shot stage game forever, they put an upper bound on the severity of punishments. When profits cannot be negative in the punishment phase, they maintain high the payoffs to cheating on the collusive agreement.

Therefore most recent papers investigate the impact of various model specifications on the sustainability of collusion with stick-and-carrot mechanisms in the style of Abreu (1986, 1988). In this category of mechanisms, if a firm deviates from collusion, all firms play a punishment strategy over one or several periods – the stick – which can be more severe than Nash reversion (i.e., it may lead to lower instantaneous profits, possibly negative, which represents a gain in realism) before returning to a collusive price or quantity. If a deviation occurs in a punishment period, the punishment phase restarts, otherwise all firms resume the collusive behavior to earn supernormal profits – the carrot. More specifically, Abreu (1986) exploits a single-period punishment mechanism for a class of repeated quantity-setting oligopoly stage games with identical sellers of a homogenous good, constant positive marginal costs, and no fixed cost. For a given discount factor, the same most severe punishment quantity firms may sell – following a deviation either from collusion or from the punishment rule, indifferently – so that firms never deviate from collusion in equilibrium, is characterized. It leads firms to reach the highest level of discounted collusive profits.

The analysis of the connection between structural conditions and collusion stability, with a stick-and-carrot mechanism à la Abreu, has been extended to many aspects. They include the case of multi-market contact (Bernheim and Whinston (1990)). Collusion is facilitated when the same firms are present on several markets. Capacity constraints have been considered (in particular Lambson (1987, 1994), Compte, Jenny, Rey (2002)). A general message is that limited and asymmetric capacities make collusion more difficult to sustain. Other papers focus on cost heterogeneity (including Rothschild (1999), Vasconcelos (2005)). It is found that collusion sustainability depends on the difference between the marginal cost levels that characterize the less and most efficient firms in the industry. Another research stream focuses on circumstances in which each firm receives a cost shock in each period (notably Athey et al. (2001, 2004, 2008)). An important result is that, when marginal costs are private information and may differ across firms, and under simple and general assumptions, ex ante cartel payoffs are maximized when firms charge the same collusive price and share the market equally, as in simpler models with complete information and symmetric firms. Other contributions, which do not always allow for the possibility of pricing below marginal costs, investigate the impact of changes in demand, with various specifications for the dynamics of shocks (see in particular Rotemberg and Saloner (1986), Haltiwanger and Harrington (1991), Bagwell and Staiger (1997)). A “tuned” collusive price gets closer to the competitive level when demand is high.¹

¹For a comprehensive survey of the literature on the factors that facilitate collusion, see Motta (2004).

Our objective is to enrich the study of the circumstances that facilitate collusion, or make it more difficult to sustain. This is done by investigating the exact role of a key assumption in the seminal paper by Abreu (1986), according to which the price is *strictly* positive for all levels of industry output, so that there is no floor for firms' losses. Hence the quantity sold – and related costs – can tend to infinity when firms charge below the marginal cost and the price approaches zero. In that case the severity of punishments, following a deviation, is unbounded. Interestingly, to our knowledge most papers – if not all – that refer to Abreu (1986, 1988) relax this assumption by introducing more structure. They either assume that demand is *finite* at all prices, or that firms have limited production capacity, so that firms' losses are bounded in a punishment period. Fudenberg and Tirole (1991) emphasize that when the severity of punishments is limited “it is not obvious precisely which actions should be specified” (p. 165) in the punishment phase, and remark that a discrete number of punishment periods does not necessarily exist that exactly compensates for the limited severity in a single period. We precisely examine this point. This is done in a setup that encompasses the main assumptions in Abreu (1986). In our model, firms sell substitutable goods (possibly differentiated), inverse demand functions are non-increasing (they can be finite at all prices), the marginal cost is constant and non-negative (it can be zero), and there can be a fixed cost. In addition to standard incentive and participation constraints, a key specification we introduce is a severity constraint, which amounts to imposing a limitation on the lowest level of profits a firm may earn. Whether the severity constraint binds or not impacts firms' choices of price or quantity in the punishment phase.

A severity constraint clearly relates to many real-world situations. It captures all contexts in which either structural conditions (i.e., demand and technological features), institutional circumstances (e.g., a price-floor regulation), or financial pressure considerations (e.g., a return on investment target), set a lower bound to firms' losses.

When demand is finite, or capacity is limited, an example of structural condition that determines the lower bound to firms' payoffs, and thereby impacts their ability to enforce severe punishment schemes, is the unit cost of production. A firm with high fixed and/or variable costs will earn more negative payoffs during aggressive pricing episodes than more efficient firms. In theory, a high unit cost will a priori have an ambiguous effect on collusion sustainability, since it also reduces incentives to deviate. On the empirical side, Symeonidis (2003) finds strong evidence that collusion is more likely in industries with high capital intensity. This result is interpreted as a consequence of high barriers to entry.² Another possible interpretation, we in-

²In Symeonidis (2003), the capital stock of the average plant, and the capital-labour ratio, are a proxy for high

investigate below, is that high unit costs – which permit severe punishments – facilitate collusion. An example of regulatory measure that reduces the severity of punishments is a price floor. As it rules out severe punishments, it should hinder collusion. In an empirical paper, Gagné et al. (2006) study the impact on prices of a price floor established by the Quebec provincial government on the retail market for gasoline. By limiting the severity of price wars, the floor was seen as a mean to reduce the ability of firms to punish retailers deviating from a high price strategy. The analysis reveals that the net effect of the floor on average price-cost margins is near zero. The impact of the floor on retail prices in low margin periods (or price wars) is actually offset by the rise in their average duration. Price wars are less severe, but they last longer.³ Our analysis offers theoretical grounds to these empirical findings. Financial parameters may also shape the severity constraint. For example, the managers of equity-dependent firms are not likely to post low operational profits for too long. The recent empirical literature has evidenced the connection between stock prices and firms’ investments, as in Baker et al. (2003). Our theoretical analysis leads us to conjecture there is also a link between financial constraints and the ability to collude.

In this paper, by delineating the largest parameter space for which a collusive strategy can be implemented, we fully characterize the conditions under which the severity constraint does reduce firms’ ability to implement a given collusive action (a price or a quantity), when the duration of punishments can be adjusted. For given cost and demand parameters, the optimal punishment path is defined as a vector of prices or quantities, played period after period, that let firms implement a given collusive strategy for the lowest admissible discount factor. When the limited severity constraint is slack, we find that the possibility to punish over several periods does *not* result in a lower threshold for the discount factor than with a single-period punishment scheme. This holds both with a slack and with a binding participation constraint (although the threshold differs across the two cases). When the severity constraint binds, the lowest discount factor for which a given collusive strategy can be implemented strictly decreases if the punishment phase is not limited to a single period. We also find that the lower bound is always reached for a *finite* number of periods. When the participation constraint is not binding, this

barriers to entry, which in turn are seen to facilitate collusion. See also Levenstein and Suslow (2006).

³The introduction of a price floor followed a price war. The local association of independent gasoline retailers reported that the price war “resulted in retail prices that were observed well below wholesale prices. It was so severe as to force several independent retailers either to close down temporarily or to exit the market” (translated from the *Mémoire de l’Association Québécoise des Indépendants du Pétrole*, June 1998, pp. 7-8). In another empirical analysis of the impact of this regulation, Houde (2008) finds that the minimum retail price floor had a significant impact on firms’ option value of staying in the market.

bound remains strictly higher than in the absence of severity constraint. Although the duration is not bounded, the severity constraint still handicaps firms' ability to collude, implying that a delayed punishment with discounting offers only an imperfect substitute for more immediate severity. When the participation constraint binds, the multi-period discount lower bound is as low as in the single-period case. When the severity constraint binds, we also find that there exists an infinity of simple punishment paths that permit firms to implement the collusive strategy. This does not hold when only incentive constraints are at play, in which case there is a *unique* optimal punishment path.

These results, which are derived in an abstract theoretical framework, are illustrated in the context of an example with specific functional forms. It helps assessing the importance of considering limited severity constraints. In a standard n -firm Cournot oligopoly model, with linear demand, horizontal product differentiation, and constant non-negative marginal costs, we investigate the circumstances in which a multi-period punishment is needed to implement optimal collusion. With no fixed cost, this is shown to depend on the number of firms, the degree of product differentiation, and the constant unit (and marginal) cost level. We obtain a partition of the parameter space into three subsets, which relate to the status of all constraints. With a linear demand (the quantity demanded is finite at all prices), the most severe punishment is obtained when the price charged by all firms is equal to zero. That is, the lower bound to punishment profits results from the non-negativity constraint in prices. Whether the severity of the endogenous single-period punishment quantity is constrained or not is shown to depend on the comparison of the marginal cost parameter with a threshold level, which in turn depends on the number of firms and on the degree of differentiation. The severity constraint can be ignored for any level of constant marginal cost and any degree of product differentiation only if there are exactly two or three firms. In that case, the results obtained in the related literature with a duopoly and a constant marginal (and unit) cost normalized to zero are robust to the introduction of a positive marginal cost. With more than three firms, we find that the marginal cost and/or the differentiation parameter must be higher than a threshold, we compute, for the limited severity constraint not to bind. This was not unveiled by past theoretical studies. Note though that, in an exploratory note, Lambertini and Sasaki (2001) already claimed that "high marginal costs tends to provide more room for tacit collusion than [...] with lower marginal costs, due to the positive price constraint" (p. 119). Moreover, introducing a positive fixed cost reduces the parameter subset in which the severity constraint binds, therefore a high unit cost facilitates collusion. This offers a more direct explanation than the one received from the

literature for the empirical observation that collusion is more likely when capital intensity is high. Also when the severity constraint binds, we establish that the lowest discount value for which collusion can be implemented decreases when the number of firms decreases, and either differentiation or the marginal cost increases. This extends results received from the literature to situations in which there is a severity constraint, and also emphasizes that all factors enhancing firms' ability to punish – in that they relax the severity constraint – facilitate collusion.

The remainder of the paper is organized as follows. Section 2 describes the model. In Section 3, we restrict the duration of a punishment phase to a single period, and identify the largest space of parameters for which a collusive strategy can be implemented. In Section 4, we obtain the main results by investigating the impact of the ability to choose a punishment action over several periods on firms' ability to collude. In section 5, the latter results are illustrated in the context of a linear Cournot model. In Section 6 we discuss our results in the light of the related literature.

2 The Model

We construct a supergame, in which identical firms in $N = \{1, \dots, n\}$ supply substitutable goods, possibly differentiated, to maximize individual intertemporal profits by simultaneously and non-cooperatively choosing a strategy a_i – or “action” – that is either a price or a quantity, in an infinitely repeated stage game over $t = 1, 2, \dots, \infty$. Each firm's action set $A \subset R_+$ is compact. The discount factor $\delta = 1/(1 + r)$, where r is the single period interest rate, is common to all firms. The continuous function $\pi_i : R_+^2 \rightarrow R_+$ relates firm i 's profits to a vector of actions $\mathbf{a} \equiv (a_i, a_{-i})$, where a_{-i} describes a symmetric action chosen by all firms in $N \setminus \{i\}$. We omit the subscript i and specify a single argument a , that is a scalar, to represent the profits $\pi(a)$ earned by firms that all choose the same action. Similarly, we denote by $\pi_i^d(a)$ the profits firm i earns when it “deviates”, in that it plays its best reply to a , as played by all other firms. The set of available actions includes a unique symmetric Nash equilibrium in pure strategy a_{NE} , implicitly defined by $\pi_i^d(a_{NE}) - \pi(a_{NE}) = 0$, all i , and a collusive action, a_m , which yields more profits (ideally a_m maximizes joint profits, a case of perfect collusion, as in the example we present in Section 5). Firms' actions may differ from period to period. An action path $\{\mathbf{a}^t\}_{t=1}^\infty$ is defined as an infinite sequence of n -dimensional vectors of actions, as chosen by each firm in each period.

We give more structure to the analysis by relating each firm i 's profits $\pi_i = p_i q_i - C(q_i)$,

where p_i is a price q_i a quantity, to the exact properties of cost and demand conditions. There are three basic assumptions:

- (A1) Firms incur a fixed cost $f \geq 0$, and a constant marginal cost $c \geq 0$, to sell substitutable goods (possibly differentiated), and their strategic variable is either price ($a = p$ in the Bertrand specification) or quantity ($a = q$ in the Cournot specification).
- (A2) Firm i 's inverse demand function $p_i : R_+^n \rightarrow R_+$ is non-increasing and continuous.
- (A3) $p_i(\mathbf{0}) > c$ and $\lim_{q_i \rightarrow \infty} p_i(q_i, \mathbf{q}_{-i}) = 0$, any \mathbf{q}_{-i} .

The main features of our model appear clearly when compared with the specifications in Abreu (1986), a reference, where the following three assumptions hold: ($\tilde{A}1$) Firms sell a homogeneous good at constant marginal cost $c > 0$, and their strategic variable is quantity; ($\tilde{A}2$) The market inverse demand function $p : R_+ \rightarrow R_+$ is *strictly* decreasing and continuous; ($\tilde{A}3$) $p(0) > c$ and $\lim_{q \rightarrow \infty} p(q) = 0$, with $q = \sum_{i \in N} q_i$. Note that the latter two assumptions imply that, for all levels of total output q , the price p is *strictly* positive. They also imply that there exists $q_c > 0$ such that $p(q_c) < c$. This says that firms can always force the price p at which firm i sells q_i down to a level strictly below c . In this case there is no floor for firms' losses since the quantity sold – and related costs – can tend to infinity when p approaches 0. The latter three assumptions are encompassed by (A1-A3). Note our assumptions also capture circumstances in which the price p_i is driven down to *exactly* zero with *finite* quantities (q_i, \mathbf{q}_{-i}) , a case that is ruled out by ($\tilde{A}1$ - $\tilde{A}3$).

As in Abreu (1986) we construct a “stick-and-carrot” penal code. All firms initially collude by choosing the collusive action a_m . If this action is played by all firms in all periods, each firm earns a discounted sum of symmetric single-period collusive profits $\pi_m \equiv \pi(a_m)$. All firms have a short-run incentive to deviate, that is to lower (increase) its own price (quantity) in order to increase individual profits at every other firm's expense. If such a deviation by one firm i in N is detected in period t , all firms switch to the punishment action a_P in period $t + 1$ (the stick). After one period of punishment, if any deviation from a_P is detected, the punishment phase restarts, otherwise all firms resume the collusive behavior by adopting the same a_m forever (the carrot). The need for a punishment is rooted in the fact that each individual firm, assuming

that all other firms play the collusive action, has a short-run incentive to lower (increase) its own price (quantity) to increase individual profits at every other firm's expense. Then the choice of a low (high) punishment price (quantity) a_P in the next period renders a free-riding behavior less attractive.

In order to express results and related proofs with notational parsimony, indifferently in the price and quantity specifications, hereafter we adopt the definition that action a' is (strictly) *more severe than* a when $\pi(a') < (=)\pi(a)$. This is denoted by $a' \prec (=)a$. A key feature of the paper is that we investigate the consequence of having a lower bound to individual punishment actions, and thereby to punishment profits. We refer to $\underline{a}_P \prec a_{NE}$ as the most severe symmetric punishment action, a parameter. Note that \underline{a}_P exists from the compactness of A . Then the continuity of π_i implies that $\underline{\pi} \equiv \pi(\underline{a}_P) < \pi(a_{NE})$ is well defined. Most realistic circumstances offer a justification for this setting. It can capture the impact of a regulatory measure. For example, a price floor will impose firms to charge above a given value (say, a wholesale price), and then will limit the severity of punishment actions (in some cases we may have $\underline{\pi} > 0$). More generally, the severity of punishments is also limited when the demanded quantity is finite at any price, including zero, for all firms. As indicated above, there is no such limit on punishments in Abreu (1986).⁴ However, we may point to such a floor in more applied and recent contributions to the literature. When the marginal cost is constant and set equal to zero, as in Häckner (1996) or Compte et al. (2002), for examples, the lowest possible profits are zero. Another example is Vasconcelos (2005), where there is a variable marginal cost and a finite demand, so that profits can be negative but limitedly so. Our more general specification also captures these cases.

We now introduce a few additional assumptions that are needed to produce formal results. The first one extends the order relationship, as follows:

(A4) If $a_{-i} \succ (=)a'_{-i}$ then $\pi_i(\mathbf{a}) > (=)\pi_i(a_i, a'_{-i})$, all $a'_{-i}, a_{-i} \preceq a_m$.

This says that, in the Bertrand (resp. Cournot) specification, firm i 's profits are non-decreasing (resp. non-increasing) with other firms' symmetric price (resp. quantity).

⁴In Abreu (1986), the action set is also compact (see Assumption (A4), p. 195). The lower bound $\bar{M}(\delta)$, in the reference paper, by definition is more severe than the optimal punishment action. To compare, in the present model, the most severe punishment $\underline{\pi}$ can be arbitrarily close to the Nash payoff $\pi(a_{NE})$.

Another specification of the model relates to deviation profits. A firm can earn positive benefits by playing its best reply to all other firms' action, only if the latter action is not too severe. Formally:

(A5) There exists $\tilde{a}_P \preceq a_{NE}$ such that $\pi_i^d(a) < (=) 0$ if and only if $a \prec (=) \tilde{a}_P$.

When all firms in $N \setminus \{i\}$ play $a \succ \tilde{a}_P$, the latter assumption implies that firm i 's *gross* deviation profits are strictly higher than the level of fixed costs, that is f . A consequence of (A5) is that $\pi(a_{NE}) \geq 0$.

Although the analysis focuses on situations with limited punishments, the latter may be very severe. A reference action that measures this severity is \hat{a}_P , which is such that the minmax profit is obtained by stopping production. We assume that:

(A6) There exists $\hat{a}_P \preceq \tilde{a}_P$ such that $\pi_i^d(a) = (>) -f$ if and only if $a \preceq (>) \hat{a}_P$.

In terms of output quantity, let $q_i^d(a)$ denote firm i 's best-reply to a , as chosen by all other firms. Assumption (A6) specifies that $q_i^d(a) = 0$ if $a \preceq \hat{a}_P$, and $q_i^d(a) > 0$ otherwise. In words, any action a , as chosen by all firms in $N \setminus \{i\}$, that is strictly more severe than \hat{a}_P , drives firm i 's profit-maximizing output (and gross profits) to zero. In particular, if $\hat{a}_P \succeq \underline{a}_P$, then the most severe symmetric punishment action, when played by all firms in $N \setminus \{i\}$, is sufficiently penalizing as to incentivize firm i to stop producing, and thereby to incur losses equal to the magnitude of fixed costs, its minmax value. Note that $\pi(a) > \underline{\pi}$ if $\hat{a}_P \succeq a \succ \underline{a}_P$, although $q_i^d(a) = q_i^d(\underline{a}_P) = 0$, with firm i 's best-reply profits $\pi_i^d(a) = \pi_i^d(\underline{a}_P) = -f \leq 0$. To gain familiarity with the notation, observe that when firms' strategic variable is price, and $c = f = \underline{\pi} = 0$, as commonly assumed for simplicity in many models, we have $\tilde{a}_P = \underline{a}_P = \hat{a}_P = 0$, a particular case.

When no constraint on the severity of a is introduced, as in most contributions to the literature, profits $\pi(a)$ are unbounded from below. In that case, since best-reply profits $\pi_i^d(a)$ do have a lower bound (a firm may always stop selling; see (A6)), we have $\pi_i^d(a) - \pi(a)$ unbounded from above. Recalling that $\pi_i^d(a_{NE}) - \pi(a_{NE}) = 0$, we know there exists at least one $\check{a} \preceq a_{NE}$ verifying $\pi_i^d(\check{a}) - \pi(\check{a}) = \pi_i^d(a_m) - \pi_m > 0$. The final assumption we introduce specifies uniqueness, for simplicity:

(A7) There exists a unique $\check{a} \prec a_m$ such that $\pi_i^d(\check{a}) - \pi(\check{a}) = \pi_i^d(a_m) - \pi_m$.

Clearly $\check{a} \prec a_{NE}$. Remark that (A7) is very mild. It captures in particular all usual situations in which the incentive to deviate $\pi_i^d(a) - \pi(a)$ increases with the severity of actions $a \preceq a_{NE}$, and also with the level of collusion $a \succ a_{NE}$.⁵

In what follows we investigate the role of the parameter $\underline{\pi}$, that is the finite lower bound to individual punishment profits, on the implementation of collusion. This is done by first considering situations in which the duration of punishments is limited to a single period.

3 Single-Period Punishments

In this section we restrict the duration of the punishment phase to a single period. For each player to have no incentive to deviate, a deviation must be followed by a punishment that leads the discounted flow of profits to be less than the actualized stream of collusive equilibrium profits. Moreover, for the punishment to be a credible threat, one should verify that firms do implement the punishment action. This occurs if individual gains to deviate from the punishment phase are smaller than the loss incurred by prolonging the punishment by one more period.⁶

Formally, the profile $\{a_m, a_P\}$, with $a_P \preceq a_m$, must satisfy two incentive constraints, we refer to hereafter as IC1 and IC2, that is

$$\pi_i^d(a_m) - \pi_m \leq \delta [\pi_m - \pi(a_P)], \quad (IC1)$$

$$\pi_i^d(a_P) - \pi(a_P) \leq \delta [\pi_m - \pi(a_P)], \quad (IC2)$$

where $\pi(a)$ denotes each firm's stage profit when all firms choose the same action a , and $\pi_i^d(a)$ is firm i 's profit from a one-shot best deviation from the action a selected by all firms in $N \setminus \{i\}$, with $a = a_m, a_P$. The first condition says that the profits associated with a deviation from the

⁵For an illustration with quantity-setting firms see Fig. 2 in Abreu (1986). In the present paper Fig. 1 is made very intuitive when a is interpreted as a price.

⁶In a trigger penal code à la Friedman (1971), a deviation implies that firms stop colluding and revert to the one-shot stage game Nash equilibrium forever. The punishment action is then self-enforcing. A stick-and-carrot setup authorizes a more severe (and also shorter) punishment phase that may lead firms to earn negative profits for some time. It is not self-enforcing, hence (IC2) is needed.

collusive action must be smaller than what is lost due to the punishment phase. The second condition says that the benefits associated with a deviation from the punishment must be smaller than the loss incurred by prolonging the punishment by one more period.

Our objective is to delineate the largest space of parameters for which the two constraints are satisfied. The problem we investigate is thus to find a punishment a_P that minimizes δ under the two incentive constraints (*IC1-IC2*). The solution a_P^* , defined as the optimal punishment, yields δ^* , the minimum. Before introducing additional constraints, we characterize a_P^* and δ^* by presenting three intermediate results.

Lemma 1. *The optimal single-period punishment action a_P^* and the discount factor lower bound δ^* are such that (*IC1*) and (*IC2*) hold with equality.*

Proof. See the appendix. ■

This first result establishes that, when $a_P = a_P^*$, and $\delta = \delta^*$, the two incentive constraints are exactly satisfied. Therefore we may compute a_P^* and δ^* by solving in (a_P, δ) the system (*IC1-IC2*) with equality signs.

To compare, recall that Abreu (1986)’s problem consists in identifying the pair of actions (a_P, a_C) that permits firms to maintain the most profitable collusive action a_C for a given discount factor δ . The two approaches are dual since the value δ^* we obtain as a solution, for a given a_m , is identical to the given value of δ that leads to the solution $a_C^* = a_m$ in Abreu’s problem. In the latter, the solution a_C^* is bounded from above by the stage-game joint-profit maximizing action. When δ is high enough for this boundary value to be implemented as a collusive equilibrium, the constraint not to deviate from collusion is slack. This says why Lemma 1 differs slightly from Abreu’s Theorem 15, in which the analogue of (*IC1*) holds with a weak inequality *only* (while the analogue to (*IC2*) holds with an equality sign, as in the present case).

Note however that the single-period punishment action that implements the collusive action, for all admissible parameter values, needs not be a_P^* . This is because a_P^* is defined as the punishment action that satisfies (*IC1-IC2*) for the *lowest* possible value of δ , that is *exactly* δ^* . When $\delta > \delta^*$, the collusive action is implementable with a “non-optimal punishment” a_P about a_P^* .

Lemma 2. Given a_m , the optimal punishment action a_P^* is such that $\pi_i^d(a_P^*) - \pi(a_P^*) = \pi_i^d(a_m) - \pi_m$.

Proof. The constraints in (IC1-IC2) can be rewritten as $\delta \geq \delta'$ and $\delta \geq \delta''$, respectively, with $\delta' \equiv [\pi_i^d(a_m) - \pi_m] / [\pi_m - \pi(a_P)]$ and $\delta'' \equiv [\pi_i^d(a_P) - \pi(a_P)] / [\pi_m - \pi(a_P)]$. Lemma 1 implies that

$$\delta^* = \delta'|_{a_P=a_P^*} = \delta''|_{a_P=a_P^*}.$$

It is then sufficient to observe that the numerators of δ' and δ'' are identically equal to conclude that the numerators $\pi_i^d(a_m) - \pi_m$ and $\pi_i^d(a_P) - \pi(a_P)$ are also equal if $a_P = a_P^*$. ■

Lemma 2 offers an implicit definition of a_P^* and says that, in the stage game, a firm's incentive to deviate from a_P^* is equal to the incentive to deviate from a_m (see Fig. 1). Note that the incentive to deviate from a_m is an upper bound to a firm's incentive to deviate from any a that verifies $a_P^* \preceq a \preceq a_m$.

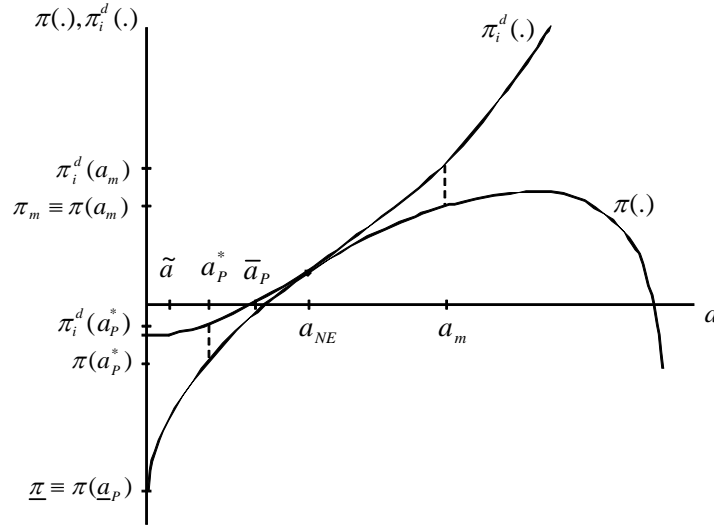


Figure 1: The optimal punishment action a_P^* is such that $\pi_i^d(a_P^*) - \pi(a_P^*) = \pi_i^d(a_m) - \pi_m$, given a_m (here with $a_P \prec \tilde{a} \prec a_P^* \prec \bar{a}_P$).

The next technical result establishes a monotonicity property.

Lemma 3. $\pi_i^d(a) \geq \pi_i^d(a')$, all $a \succeq a'$.

Proof. Recall that $\pi_i^d(a_{-i}) \equiv \pi_i(a_i^d(a_{-i}), a_{-i})$, with $a_i^d(a_{-i}) \equiv \arg \max_{a_i} \pi_i(a_i, a_{-i})$. From the definition of $a_i^d(a_{-i})$, we have

$$\pi_i(a_i^d(a_{-i}), a_{-i}) \geq \pi_i(a_i^d(a'_{-i}), a_{-i}),$$

all a_{-i}, a'_{-i} . Next, from the monotonicity of $\pi_i(a_i^d(a'_{-i}), a_{-i})$ in a_{-i} , we know that $a_{-i} \succeq a'_{-i}$ implies

$$\pi_i(a_i^d(a'_{-i}), a_{-i}) \geq \pi_i(a_i^d(a'_{-i}), a'_{-i}).$$

This leads to $\pi_i^d(a_{-i}) \geq \pi_i^d(a'_{-i})$ by transitivity. ■

We now introduce two additional constraints. The first one is a participation constraint.⁷ It specifies that each firm, when it actualizes the future stream of profits earned from the period of punishment onward, must be incentivized to continue playing the game even if it earned negative profits for a while. Formally, it must be the case that $\pi(a_P) + \sum_{k=1}^{\infty} \delta^k \pi_m \geq 0$. A simple reorganization of terms, toward a more intuitive expression, leads to

$$(1 - \delta) [\pi_m - \pi(a_P)] \leq \pi_m. \quad (PC)$$

In words, the participation constraint is satisfied when the profit a firm forgoes in the punishment period, that is the difference $\pi_m - \pi(a_P)$, is not greater than the discounted stream of collusive profits earned in all following periods, that is $\pi_m / (1 - \delta)$.⁸ Note that (IC2), we may rewrite as $(1 - \delta) [\pi_m - \pi(a_P)] \leq \pi_m - \pi_i^d(a_P)$, can be easily compared to (PC). Recalling from (A5) that $\pi_i^d(a_P) > (=) 0$ if and only if $a_P \succ (=) \tilde{a}_P$, observe that (IC2) is (weakly) stronger than (PC) if and only if $a_P \succ (=) \tilde{a}_P$. It follows that, when $a_P^* \prec \tilde{a}_P$, (PC) is violated, hence δ^* is not attainable. In this case, toward a solution to the participation-constrained problem we define a particular punishment action, denoted by \bar{a}_P , that satisfies exactly both (IC1) and (PC). In formal terms, $\pi(\bar{a}_P) = \pi_m - \pi_i^d(a_m)$.⁹ For notational clarity, let $\bar{\pi} \equiv \pi(\bar{a}_P)$.

⁷In a price-setting oligopoly with a homogenous good, a constant average cost, and capacity constraints, Lambson (1987) introduces it as an individual rationality constraint.

⁸When compared with a trigger penal code (where a deviation implies that firms stop colluding and revert to the one-shot stage game Nash equilibrium forever, à la Friedman (1971)), a stick-and-carrot setup authorizes a shorter and more severe punishment phase that may lead firms to earn negative profits for some time without violating the individual rationality constraint (PC). This results in a more efficient punishment in the sense that collusion can be sustained for lower values of the discount factor.

⁹The implicit definition of \bar{a}_P obtains by rewriting (IC1) as $\delta \geq [\pi_i^d(a_m) - \pi_m] / [\pi_m - \pi(a_P)]$, and (PC) as $\delta \geq -\pi(a_P) / [\pi_m - \pi(a_P)]$. Then observe that the denominators are equal. Note that $\bar{a}_P \prec a_{NE}$ since $\pi(\bar{a}_P) < 0$.

A useful technical result is:

Lemma 4. $\bar{a}_P \succeq a_P^*$ if and only if $\tilde{a}_P \succeq a_P^*$.

Proof. Sufficiency: If $\tilde{a}_P \succeq a_P^*$ then $\pi_i^d(a_P^*) \leq 0$ by (A5). Suppose $\bar{a}_P \prec a_P^*$ and look for a contradiction. By Lemma 3, $\bar{a}_P \prec a_P^*$ implies $\pi_i^d(\bar{a}_P) \leq \pi_i^d(a_P^*)$ hence $\pi_i^d(\bar{a}_P) \leq 0$. It also implies by Lemma 2 and (A7) that $\pi_i^d(\bar{a}_P) - \bar{\pi} > \pi_i^d(a_m) - \pi_m$. It follows that $\bar{\pi} < \pi_m - \pi_i^d(a_m) + \pi_i^d(\bar{a}_P) \leq \pi_m - \pi_i^d(a_m)$, which clearly contradicts the definition of \bar{a}_P . As a result $\tilde{a}_P \succeq a_P^*$ implies $\bar{a}_P \succeq a_P^*$.

Necessity: If $\tilde{a}_P \prec a_P^*$, suppose that $\bar{a}_P \succeq a_P^*$ and look for a contradiction. Clearly $\bar{\pi} < 0$ implies $\bar{a}_P \prec a_m$. By Lemma 2 and (A7), $a_P^* \preceq \bar{a}_P \prec a_m$ implies that $\pi_i^d(\bar{a}_P) - \bar{\pi} \leq \pi_i^d(a_m) - \pi_m$. From the very definition of \bar{a}_P , it follows that $\pi_i^d(\bar{a}_P) \leq 0 = \pi_i^d(\tilde{a}_P)$. By Lemma 3, this implies that $\bar{a}_P \preceq \tilde{a}_P$ and by transitivity through $\tilde{a}_P \prec a_P^*$, that $\bar{a}_P \prec a_P^*$, a contradiction. As a result $\bar{a}_P \succeq a_P^*$ implies $\tilde{a}_P \succeq a_P^*$. ■

The next constraint is central to the analysis. It imposes a limit to the severity of the symmetric punishments all firms may inflict on each other in a single period. When punishment profits may not be infinite, a_P must satisfy

$$\pi(a_P) \geq \underline{\pi}. \quad (SC)$$

This can be rooted in structural conditions (i.e., the demanded quantity can be specified to be finite at any price, including zero), or in institutional features (e.g., a regulation). In what follows we refer to this weak inequality as the severity constraint. It does not appear in Abreu (1986)'s seminal paper, where the inverse demand is *strictly* monotonic, so that below-cost pricing may result in infinite losses to all firms. The order relation on the set of punishment actions a_P , as defined in the previous section, implies that (SC) can be rewritten equivalently as $a_P \succeq \underline{a}_P$. This does not mean that the optimal a_P^* , when it satisfies (SC), cannot not be associated with very low profits in the period of punishment. Recall from (A5) that the “lower” bound \underline{a}_P , when played by all firms in $N \setminus \{i\}$, can be sufficiently severe as to incentivize firm i to stop producing as a best-reply.

We may now write the δ -minimization problem in a_P as follows:

$$\begin{aligned} \min_{a_P} \quad & \delta \\ \text{s.t.} \quad & IC1; IC2; PC; SC \end{aligned} \quad (1)$$

We now determine the lowest δ for which the collusive action a_m is implementable. This relies on the comparison of the structurally defined punishment actions a_P^* , \bar{a}_P , and \underline{a}_P .

Proposition 1. *The collusive action a_m is implementable with a single-period punishment if and only if $\delta \geq \delta_1^*$, with*

$$\delta_1^* = \begin{cases} \delta^* \equiv \frac{\pi_i^d(a_m) - \pi_m}{\pi_m - \pi(a_P^*)} & \text{if } a_P^* \succeq \underline{a}_P, \bar{a}_P \quad (\text{regime 1}); \\ \bar{\delta} \equiv \frac{\pi_i^d(a_m) - \pi_m}{\pi_m - \bar{\pi}} & \text{if } \bar{a}_P \succeq \underline{a}_P, a_P^* \quad (\text{regime 2}); \\ \underline{\delta} \equiv \frac{\pi_i^d(a_m) - \pi_m}{\pi_m - \underline{\pi}} & \text{if } \underline{a}_P \succeq a_P^*, \bar{a}_P \quad (\text{regime 3}). \end{cases} \quad (2)$$

Proof. First we solve a less constrained version of (1), in which (PC) and (SC) are absent. Then we reintroduce each of the latter two constraints separately. (See the appendix). ■

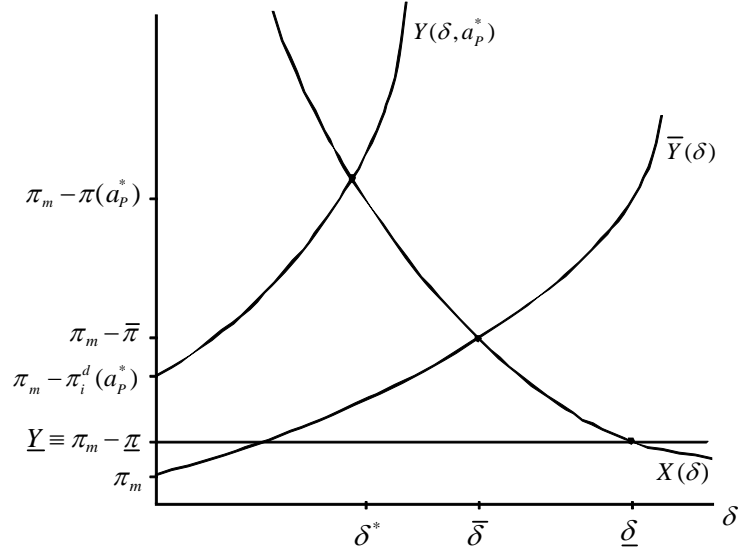


Figure 2: The two ICs in $(IC1-IC2)$ rewrite $X(\delta) \leq \pi_m - \pi(a_P) \leq Y(\delta, a_P)$, with $X(\delta) \equiv [\pi_i^d(a_m) - \pi_m] / \delta$ and $Y(\delta, a_P) \equiv [\pi_m - \pi_i^d(a_P)] / (1 - \delta)$. When PC and SC are absent, the optimal punishment a_P^* and the threshold δ^* are such that $X(\delta^*, a_P^*) = Y(\delta^*)$. Here $a_P^* \prec \bar{a}_P \prec \underline{a}_P$, therefore SC binds. The severity-constrained optimal punishment is \underline{a}_P , and firms may implement a_m for all $\delta \geq \underline{\delta}$.

The three regimes identified in Proposition 1 reflect which constraint is at play in the δ -minimization problem (1). In regime 1, the two incentive constraints are stronger than (PC) and (SC) . The optimal punishment is a_P^* , and the minimized discount factor is $\delta_1^* = \delta^*$ (here the subscript “1” refers to the single-period punishment case). In regime 2, $(IC1)$ and (PC) bite,

the optimal punishment is \bar{a}_P , and a_m can be implemented only if $\delta \geq \delta_1^* = \bar{\delta}$; while in regime 3, $(IC1)$ and (SC) are binding, the optimal punishment is \underline{a}_P , and a_m can be implemented only if $\delta \geq \delta_1^* = \underline{\delta}$. Note that $(IC1)$ is active in all regimes. In fact a firm's incentive to deviate from the collusive action, as compared to what is lost in the punishment phase, remains the same in the three regimes. What changes across regimes is the loss firms incur in the punishment phase, hence the incentive to deviate (see $(IC2)$) or to participate (see (PC)), together with the range of possible punishments (see (SC)). An important point is that the comparison between regime 1 and 2 differs in kind from the comparison between regime 3 and either regime 1 or 2. More precisely, whether a solution is in regime 1 or 2 depends on whether (PC) is stronger than $(IC2)$ or not. This is a consequence rooted in the firms' payoff functions. Whether regime 3 arises or not can depend also on the strategy set, which may be limited "from below" for all sorts of institutional or financial reasons that do not relate to cost or demand conditions.

Remark 1. If $a_P^* \succeq \underline{a}_P, \bar{a}_P$, so that regime 1 applies, $\delta^* \geq \bar{\delta}, \underline{\delta}$.

This remark emphasizes a subtle aspect of Proposition 1. Obviously, when either regime 2 or 3 applies, so that either (PC) or (SC) binds, respectively, we have $\delta^* \leq \bar{\delta}, \underline{\delta}$. Indeed the δ -minimization problem (1) is more constrained than when only the incentive constraints $(IC1)$ and $(IC2)$ are considered. However, when regime 1 applies, $(IC1)$ and $(IC2)$ are stronger than both (PC) and (SC) . Hence the relevant threshold δ^* cannot be lower than $\bar{\delta}$ and $\underline{\delta}$. In fact, in the single-period punishment problem, at most two constraints may bind, that determine the threshold for δ .

Recalling that our objective is to identify the largest space of parameters for which a given collusive action is implementable, it remains to investigate the possibility to lengthen the duration of the punishment phase. The intuition is that, by shifting to a multi-period punishment scheme, firms may penalize more severely a deviation than in the single-period framework. This may soften the lower bound condition on the discount factor, and thus facilitate collusion.¹⁰ We tackle this next.

¹⁰Several periods of punishment have been considered only in a few theoretical contributions with more specific assumptions than in the present model. Lambson (1987) considers price-setting sellers of a homogenous good, a constant average cost, with capacity constraints. Häckner (1996) constructs a repeated price-setting duopoly model, with spatial differentiation, and a constant average cost normalized to zero. In Lambertini and Sasaki (2002), again there are two firms and a constant marginal average cost, but with another specification of the horizontal differentiation assumption, together with a non-negative constraint on quantities, but not on prices.

4 Multi-Period Punishments

In this section we introduce the possibility for firms to choose a punishment action over several periods. The objective is to investigate the impact of the extended length of punishment on firms' ability to implement collusion, when the severity of punishment is limited in each period.

To do that, consider a stick-and-carrot penal code in which, if any deviation from a_m by any firm is detected, all firms switch to a l -period punishment phase (the stick) during which they play $a_{P,k}$, with $k = 1, \dots, l$. Punishment actions may vary from one period to another. A deviation from the punishment action may occur in any period of punishment. If this occurs, the punishment phase restarts for l more periods, after which all firms revert to the initial collusive action a_m forever (the carrot).

Formally, the two incentive constraints (*IC1*) and (*IC2*) are now extended to

$$\pi_i^d(a_m) + \sum_{k=1}^l \delta^k \pi(a_{P,k}) + \sum_{k=l+1}^{\infty} \delta^k \pi_m \leq \sum_{k=0}^{\infty} \delta^k \pi_m, \quad (3)$$

and

$$\begin{aligned} & \sum_{k=1}^{s-1} \delta^k \pi(a_{P,k}) + \delta^s \pi_i^d(a_{P,s}) + \sum_{k=s+1}^{s+l} \delta^k \pi(a_{P,k-s}) + \sum_{k=s+l+1}^{\infty} \delta^k \pi_m \\ & \leq \sum_{k=1}^l \delta^k \pi(a_{P,k}) + \sum_{k=l+1}^{\infty} \delta^k \pi_m, \end{aligned} \quad (4)$$

respectively, for any period s in which a firm deviates from the penal code, with $1 \leq s \leq l$ (we adopt the convention that $\sum_{k=1}^{s-1} \delta^k \pi(a_{P,k}) = 0$ if $s = 1$), all i .

Given a_m , the vector $\mathbf{a}_P \equiv (a_{P,1}, \dots, a_{P,l})$ is an equilibrium of the supergame if and only if (3) and (4) are satisfied. There are $1 + l$ incentive constraints in all: the single constraint in (3) says that the gain earned by deviating from the collusive action must be smaller than what is lost over the l periods of punishment; the other l constraints in (4) say that the gain to deviate from the punishment phase, in any period s , with $1 \leq s \leq l$, must be smaller than the loss incurred by reinitiating the punishment phase.

To simplify the presentation of incentive constraints and clarify their interpretation, we now introduce a value function. If a firm does *not* deviate from the punishment path, the continuation

profits it earns from period $s + 1$ onward is

$$V_s(\mathbf{a}_P, \delta) = \sum_{k=s+1}^l \delta^{k-s-1} \pi(a_{P,k}) + \sum_{k=l+1}^{\infty} \delta^{k-s-1} \pi_m. \quad (5)$$

Here $s = 0$ indicates that the l -period flow of punishment profits is not truncated from below, whereas $s = l$ means that exactly all punishment profits are removed, so that only collusive profits are considered from period $l + 1$ onward. Note from (5) that $a_{P,l+1} = a_m$ implies $V_s(\mathbf{a}_P, \delta) \leq V_l(\mathbf{a}_P, \delta) = \pi_m / (1 - \delta)$, all s . This also implies that $V_l(\mathbf{a}_P, \delta) = V_0(\mathbf{a}_m, \delta)$.

Then the multi-period incentive constraints in (3) and (4) are

$$\pi_i^d(a_m) - \pi_m \leq \delta [V_0(\mathbf{a}_m, \delta) - V_0(\mathbf{a}_P, \delta)], \quad (MIC1)$$

and

$$\pi_i^d(a_{P,1}) - \pi(a_{P,1}) \leq \delta [V_1(\mathbf{a}_P, \delta) - V_0(\mathbf{a}_P, \delta)], \quad (MIC2)$$

...

$$\pi_i^d(a_{P,s}) - \pi(a_{P,s}) \leq \delta [V_s(\mathbf{a}_P, \delta) - V_0(\mathbf{a}_P, \delta)], \quad (\dots)$$

...

$$\pi_i^d(a_{P,l}) - \pi(a_{P,l}) \leq \delta [V_l(\mathbf{a}_P, \delta) - V_0(\mathbf{a}_P, \delta)], \quad (MICl+1)$$

respectively, with $1 \leq s \leq l$. Note that $\pi(a_{P,s}) \leq \pi_i^d(a_{P,s})$ requires that $V_0(\mathbf{a}_P, \delta) \leq V_s(\mathbf{a}_P, \delta)$, all s , a feasibility condition of the punishment scheme.

In (MIC 1), that is the first multi-period punishment incentive constraint, we compare a firm's payoffs when it colludes by choosing a_m , that is $\pi_m + \delta V_0(\mathbf{a}_m, \delta)$, with the payoffs it earns by deviating, that is $\pi_i^d(a_m) + \delta V_0(\mathbf{a}_P, \delta)$. It is individually rational to stick to the collusive action if this first constraint is satisfied. The next incentive constraints, one for each period of punishment, compare a firm's payoff when it implements a punishment action, with the payoffs it earns by deviating. Most precisely, in (MIC 2), the second multi-period punishment incentive constraint compares the firm's payoff when it plays $a_{P,1}$, that is $\pi(a_{P,1}) + \delta V_1(\mathbf{a}_P, \delta)$, with the payoffs it earns by deviating, that is $\pi_i^d(a_{P,1}) + \delta V_0(\mathbf{a}_P, \delta)$. The next row describes the same comparison for the next period of punishment, and so on, down to (MIC l+1). A firm will not deviate from the l -period punishment path if all constraints of rank $s = 1, \dots, l$ are satisfied.

A first technical claim is a multi-period counterpart to Lemma 1, as offered above in the single-period punishment case.

Lemma 5. *Given $a_{P,1}$, the lowest discount factor δ verifying (MIC 1-MIC 2) results from punishment actions $a_{P,k}$, with $k > 1$, such that the two multi-period incentive constraints bind.*

Proof. See the appendix. ■

The multi-period participation constraint is $V_0(\mathbf{a}_P, \delta) \geq 0$. In words, the continuation profits, from the first period of punishment onward, must remain non-negative for a firm to implement the punishment \mathbf{a}_P . Interestingly this can also be rewritten as

$$(1 - \delta) [V_0(\mathbf{a}_m, \delta) - V_0(\mathbf{a}_P, \delta)] \leq \pi_m, \quad (MPC)$$

an intuitive generalization of the single-punishment period counterpart in (PC). This says that the sum of profits each firm foregoes by implementing the l -period punishment \mathbf{a}_P , that is the difference $V_0(\mathbf{a}_m, \delta) - V_0(\mathbf{a}_P, \delta)$, cannot be more than the discounted stream of profits earned in all collusive periods that follow, $\pi_m / (1 - \delta)$.¹¹

Observe from (MIC 1 and (MPC) that the value differential $V_0(\mathbf{a}_m, \delta) - V_0(\mathbf{a}_P, \delta)$ is bounded from below by $[\pi_i^d(a_m) - \pi_m] / \delta$ and from above by $\pi_m / (1 - \delta)$, respectively. This yields:

Lemma 6. *(MIC 1) and (MPC) are compatible only if $[\pi_i^d(a_m) - \pi_m] / \delta \leq \pi_m / (1 - \delta)$, that is $\delta \geq [\pi_i^d(a_m) - \pi_m] / \pi_i^d(a_m) = \bar{\delta}$.*

Proof. The threshold $\delta = [\pi_i^d(a_m) - \pi_m] / \pi_i^d(a_m)$ follows directly from the comparison of the left-hand side and right-hand side of (MIC 1 and (MPC), respectively. This threshold does not differ from $\bar{\delta}$, as introduced in Proposition 1, since the denominator $\pi_i^d(a_m) = \pi_m - \bar{\pi}$ from the implicit definition of \bar{a}_P . ■

Therefore there can be no l -period punishment \mathbf{a}_P that implements a_m when the discount factor is strictly lower than $\bar{\delta}$. This means that the lengthening of the punishment scheme cannot help relaxing the participation constraint.

¹¹The latter interpretation of (MPC) is even more intuitive when one sees that $V_0(\mathbf{a}_M, \delta) - V_0(\mathbf{a}_P, \delta) = \sum_{k=1}^l \delta^{k-1} (\pi(a_M) - \pi(a_{P,k}))$, so that $l = 1$ leads to (PC), the participation constraint in the single-period punishment setup.

Now the multi-period severity constraint is

$$\pi(a_{P,k}) \geq \underline{\pi}, \quad (MSC)$$

with $1 \leq k \leq l$, all $l \geq 2$. In words, (MSC) captures structural conditions imposing that, in any period k of the punishment phase, a firm's profit cannot be driven below $\underline{\pi}$, a parameter.

The multi-period punishment problem is then

$$\begin{aligned} \min_{(a_{P,1}, \dots, a_{P,l})} \quad & \delta \\ \text{s.t.} \quad & (MIC\ 1 - MIC\ l+1); MPC; MSC \end{aligned} \quad (7)$$

For any given l , the optimal multi-period punishment is the solution in $\mathbf{a}_P = (a_{P,1}, \dots, a_{P,l})$ to (7). It yields the lowest possible value of the discount factor, we denote by δ_l^* , that authorizes firms to implement a_m , under all constraints. In what follows we examine successively the role of the $1 + l$ multi-period incentive constraints $(MIC\ 1 - MIC\ l+1)$, the participation constraint (MPC) , and the severity constraint (MSC) .

We now establish that, in the absence of participation and severity constraints, the possibility to punish over several periods does *not* result in an optimal punishment path that differs from the single-period punishment case.

Proposition 2. *In the multi-period punishment scheme, if $a_P^* \succeq \bar{a}_P, \underline{a}_P$ the collusive action a_m is implementable if and only if $\delta \geq \delta^*$, and $\mathbf{a}_P^* \equiv (a_P^*, a_m, \dots, a_m)$ is optimal.*

Proof. There are two steps (see the appendix):

- (i) We investigate a less constrained version of (7) by leaving aside the last $l - 1$ multi-period incentive constraints, together with (MPC) and (MSC) , to keep only $(MIC\ 1)$ and $(MIC\ 2)$. This is done by capitalizing on Lemma 5: we solve in (δ, V_1) the system $(MIC\ 1 - MIC\ 2)$ with equality signs, to obtain $(\delta^*(a_{P,1}), V_1(\mathbf{a}_P, \delta^*(a_{P,1})))$; then we identify the level of $a_{P,1}$ that minimizes $\delta^*(a_{P,1})$ under the feasibility constraint that $V_1(\mathbf{a}_P, \delta^*(a_{P,1})) \leq V_l(\mathbf{a}_P, \delta^*(a_{P,1})) = \pi_m / (1 - \delta^*(a_{P,1}))$. This leads to the minimizer $a_{P,1}^* = a_P^*$, and to the discount factor $\delta^*(a_{P,1}^*) = \delta^*$.
- (ii) We show that $(\delta^*, V_1(\mathbf{a}_P, \delta^*(a_P^*)))$ satisfies all incentive constraints in $(MIC\ 1 - MIC\ l+1)$ as well as $(MPC - MSC)$. ■

When (MPC) and (MSC) are slack, by playing in the first period the action obtained in the previous section as a (unique) solution to the δ -minimization problem with a *single*-period punishment scheme (i.e., $a_{P,1}^* = a_P^*$), followed in all $l-1$ subsequent periods by the same collusive action (i.e., $a_{P,k}^* = a_m$, all $k = 2, \dots, l$), one obtains the lowest possible value of δ for which the collusive action a_m is implementable. The threshold value of the discount factor we obtain in this l -period punishment scheme is the *same* as in the single-punishment case, namely δ^* .

Remark 2. If $a_P^* \succeq \bar{a}_P, \underline{a}_P$ there is a unique punishment path \mathbf{a}_P^* that permits firms to implement a_m for $\delta = \delta^*$.

In other words, as long as the participation and severity constraints are not binding, there is one best way to solve (7). In a supgame with discounting, late punishments have less impact. Firms must charge a low price or supply a large quantity as early as possible, that is in the first punishment period, in order to minimize the discount factor at which a_m is implementable.

Next, we establish that, when the multi-period participation constraint binds, again the possibility to punish over several periods does *not* enlarge the space of parameters for which the collusive action is implementable.

Proposition 3. In the multi-period punishment scheme, if $\bar{a}_P \succeq \underline{a}_P, a_P^*$, the collusive action a_m is implementable if and only if $\delta \geq \bar{\delta}$, and $\bar{\mathbf{a}}_P \equiv (\bar{a}_P, a_m, \dots, a_m)$ is optimal.

Proof. There are two steps (see the appendix):

- (i) In addition to $(MIC\ 1)$ and $(MIC\ 2)$, we introduce (MPC) in the less constrained version of (7) where the last $l-1$ multi-period incentive constraints and (MSC) are left aside. We show that (MPC) is stronger than $(IC2)$ if $a_P^* \preceq \tilde{a}_P$. Then a_m is implementable with the l -period punishment $\bar{\mathbf{a}}_P \equiv (\bar{a}_P, a_m, \dots, a_m)$ if $\delta = \bar{\delta}$, that is the lower bound to the interval of δ for which $(MIC\ 1)$ and (MPC) are compatible.
- (ii) We obtain that $(\bar{\delta}, \bar{\mathbf{a}}_P)$ satisfies all other incentive constraints $(MIC\ 3-MIC\ l+1)$, in which case $\bar{\delta}$ is a solution to (7) and $\bar{\mathbf{a}}_P$ is optimal. ■

When (MPC) binds, by playing \bar{a}_P in the first punishment period (as in the single-period scheme), followed by the same collusive action (i.e., $\bar{a}_{P,k} = a_m$, all $k = 2, \dots, l$) in all $l-1$ subsequent periods, one obtains the lowest possible value of δ for which the collusive action a_m

is implementable, namely $\bar{\delta}$, the same as in the single-punishment case when (PC) binds. The intuition for this result is straightforward. Indeed the participation constraint determines the maximum *total* punishment a firm can incur (as opposed to a per-period punishment). In fact the constraint is the same in the single- and multi-period schemes, since the definition of the maximum total punishment does not depend on the number of periods. When the participation constraint binds with only one punishment period, it cannot be relaxed by extending the number of periods.

Remark 3. If $\bar{a}_P \succ a_P^*$ there is an infinite number of punishments that permit firms to implement a_m for $\delta = \bar{\delta}$.

Proof. See the appendix. ■

This says that, when (MPC) binds, the punishment $\bar{\mathbf{a}}_P \equiv (\bar{a}_P, a_m, \dots, a_m)$ is only one way, among others, of implementing a_m when the discount factor is the lowest possible, at $\bar{\delta}$. Firms may opt for a softer first-period action if they choose to lengthen the punishment phase to one or several subsequent periods, before reverting to a_m . While the possibility to punish over several periods does *not* permit firms to reduce the discount factor threshold for which the collusive action is implementable, the space of policies that allow them to do so is strictly larger than in the single-period punishment case.

We now turn to the case of a binding severity constraint. We shall see that it differs qualitatively from the previous cases, in that additional punishment periods result in a strictly lower discount threshold than with a single-period scheme. To show this, we first need the following two technical results:

Lemma 7. The lowest δ compatible with $(MIC\ 1)$ - $(MIC\ 2)$ and (MSC) is $\underline{\delta}' \equiv \frac{\pi_i^d(a_m) - \pi_m}{\pi_i^d(a_m) - \pi_i^d(\underline{a}_P)}$. If $\underline{a}_P \succeq a_P^*, \bar{a}_P$, for collusion to be implemented at $\delta = \underline{\delta}'$, it must be the case that the two constraints $(MIC\ 1)$ and $(MIC\ 2)$ are binding and that $a_{P,1} = \underline{a}_P$.

Proof. As for the proof of Proposition 2 consider the δ -minimization problem with the constraints $(MIC\ 1)$ and $(MIC\ 2)$ only. Recall that (15) must hold with an equality sign throughout for δ to be minimized (Lemma 5), and the solution in (δ, V_1) is $(\delta^*(a_{P,1}), V_1(\mathbf{a}_P, \delta^*(a_{P,1})))$, with

$$\delta^*(a_{P,1}) = \frac{\pi_i^d(a_m) - \pi_m}{\pi_i^d(a_m) - \pi_i^d(a_{P,1})}.$$

The monotonicity of $\pi_i^d(a_{P,1})$ in $a_{P,1}$ (Lemma 3) implies that $\delta^*(a_{P,1})$ is monotone non-decreasing in $a_{P,1}$. Then substitute \underline{a}_P for $a_{P,1}$ to find $\delta^*(\underline{a}_P) = \underline{\delta}'$. ■

Lemma 8. *For all \underline{V} verifying $\underline{\pi} < (1 - \delta)\underline{V} \leq \pi_m$, there exists a finite l and a punishment $\underline{\mathbf{a}}_P \equiv (\underline{a}_P, a_{P,2}, \dots, a_{P,k}, \dots, a_{P,l})$, with $a_{P,k} \succeq \underline{a}_P$ for all $k > 1$, such that $V_1(\underline{\mathbf{a}}_P, \delta) = \underline{V}$.*

Proof. There are three steps (we develop in the appendix): (1) we show that, given any δ , for any $l \geq 2$ there exists a punishment $\underline{\mathbf{a}}_P^l$ of length l such that $V_1(\underline{\mathbf{a}}_P^l, \delta) = \underline{V}$ for any \underline{V} in a closed interval I_l we define; (2) we establish that the upper-bound of I_{l+1} is the lower bound of I_l so that their finite union $I_L = \cup_{l=1}^L I_l$ is itself a closed interval; (3) we conclude by evidencing that the lower and upper bounds of the union are respectively $\underline{\pi}/(1 - \delta)$ and $\pi_m/(1 - \delta)$. ■

The next proposition describes the optimal punishment, and characterizes the associated discount threshold, when (MSC) binds.

Proposition 4. *In the multi-period punishment scheme, if $\underline{a}_P \succeq a_P^*, \bar{a}_P$, there exists a finite $l > 1$ such that the collusive action a_m is implementable if and only if $\delta \geq \sup\{\underline{\delta}', \bar{\delta}\} \equiv \underline{\delta}_M$.*

Proof. There are two cases (we develop in the appendix) that depend on the comparison of $\underline{\delta}'$ and $\bar{\delta}$. In both cases: (1) we establish that there exists a finite punishment, we denote \mathbf{a}_P , which is such that $V_1(\mathbf{a}_P, \delta)$ is equal to a particular value we explicit; (2) we check that all incentive constraints are satisfied; (3) we also verify that the participation and severity constraints hold. ■

Remark 4. *If (MSC) is strictly binding, that is if $\underline{a}_P \succ a_P^*, \bar{a}_P$, there exists a continuum of finite optimal punishments $(\underline{a}_P, a_2, \dots, a_l)$, with $l \geq 2$, such that a_m is implementable for $\delta = \underline{\delta}_M$.*

Proof. Consider the punishment profile of Lemma 8, that is $\mathbf{a}_P^l \equiv (a_{P,1}^l, a_{P,2}^l, \dots, a_{P,k}^l, \dots, a_{P,l}^l)$, where $a_{P,k}^l = \underline{a}_P$ for all $k = 1, 2, \dots, l - 1$, and $a_{P,l}^l \succeq \underline{a}_P$. We know from Proposition 4 that there exists a punishment profile of this kind that allows firms to implement a_m for $\delta = \underline{\delta}_M$. We also have shown that, for this punishment profile, the $(MIC\ l + 1)$ constraint holds and is slack. One may construct a $l + 1$ period punishment profile identical to \mathbf{a}_P^l up to the period $k = l - 1$ and with $a'_{P,l} \succ a_{P,l}^l$ and $a'_{P,l+1} \prec a_m$ such that

$$\pi(a_{P,l}) + \delta\pi_m = \pi(a'_{P,l}) + \delta\pi(a'_{P,l+1})$$

and all incentive constraints are satisfied. ■

In the next final proposition, that synthesizes the previous results, we may now rank all the discount thresholds introduced above.

Proposition 5.

If $\underline{a}_P \succ a_P^, \bar{a}_P$ then either $\bar{a}_P \preceq a_P^*$ so that $\delta^* < \underline{\delta}_M < \underline{\delta}$, or $\bar{a}_P \succ a_P^*$ in which case $\bar{\delta} \leq \underline{\delta}_M < \underline{\delta}$. We have $\underline{\delta}_M = \bar{\delta}$ if and only if $\tilde{a}_P \succeq \underline{a}_P \succ \bar{a}_P \succ a_P^*$.*

Proof. We assume that $\underline{a}_P \succ a_P^*, \bar{a}_P$. To see that $\underline{\delta}_M < \underline{\delta}$, recall that $\underline{\delta}_M \equiv \sup\{\underline{\delta}', \bar{\delta}\}$ and consider the two possible cases: (i) If $\underline{\delta}_M = \bar{\delta}$ then it suffices to recall that $\underline{a}_P \succ a_P^*, \bar{a}_P$ implies $\underline{\delta} > \bar{\delta}$ (see Remark 1) to conclude. (ii) If $\underline{\delta}_M = \underline{\delta}'$ then compare the expressions of the denominators of $\underline{\delta}'$ and $\underline{\delta}$. We have $\pi_i^d(a_m) - \pi_i^d(\underline{a}_P) > \pi_m - \underline{\pi}$ if and only if $\pi_i^d(a_m) - \pi_m > \pi_i^d(\underline{a}_P) - \underline{\pi}$. Then it suffices to see that the latter inequality is implied by $\delta^* < \underline{\delta}$, which is a consequence of $\underline{a}_P \succ a_P^*, \bar{a}_P$ (again see Remark 1).

Next, suppose first that $\bar{a}_P \preceq a_P^*$. Then $\delta^* < \underline{\delta}_M$ follows directly from Remark 2. Second, suppose that $\bar{a}_P \succ a_P^*$. Then $\bar{\delta} \leq \underline{\delta}_M$ follows directly from the definition of $\underline{\delta}_M$. Then to demonstrate that $\bar{\delta} = \underline{\delta}_M$ iff $\tilde{a}_P \succeq \underline{a}_P \succ \bar{a}_P \succ a_P^*$, note first from Assumption (A5) that $\tilde{a}_P \succeq \underline{a}_P$ if and only if $\pi_i^d(\underline{a}_P) \leq 0$, implying that $\pi_i^d(a_m) - \pi_i^d(\underline{a}_P) \geq \pi_i^d(a_m)$. Since $\bar{\pi} = \pi_m - \pi_i^d(a_m)$ by definition, we have $\bar{\delta} \equiv \frac{\pi_i^d(a_m) - \pi_m}{\pi_m - \bar{\pi}} = \frac{\pi_i^d(a_m) - \pi_m}{\pi_i^d(a_m)}$. Hence $\pi_i^d(a_m) - \pi_i^d(\underline{a}_P) \geq \pi_i^d(a_m)$ implies that $\bar{\delta} \equiv \frac{\pi_i^d(a_m) - \pi_m}{\pi_i^d(a_m)} \geq \underline{\delta}' \equiv \frac{\pi_i^d(a_m) - \pi_m}{\pi_i^d(a_m) - \pi_i^d(\underline{a}_P)}$, implying that $\underline{\delta}_M = \bar{\delta}$ (by definition), as needed. ■

This result establishes that, when $\underline{a}_P \succ a_P^*, \bar{a}_P$, and additional punishment periods are introduced, the lowest discount factor $\underline{\delta}_M$ that permits the implementation of a_m cannot be as low as δ^* , and can attain $\bar{\delta}$ only in particular circumstances. In other words, when regime 3 applies in the single-period scheme, a delayed punishment with discounting offers an imperfect substitute for more immediate severity. More precisely, suppose that, absent the (multi-period) severity constraint (*MSC*), regime 1 would apply. Then recall from Remark 2 that the only punishment profile allowing firms to implement collusion when $\delta = \delta^*$, a lower bound, is $\mathbf{a}_P^* \equiv (a_P^*, a_m, \dots, a_m)$. Introducing a severity constraint such that regime 3 applies obviously does not help, since it makes a_P^* unattainable in the first punishment period. In that case a longer punishment phase permits firms to increase the total punishment, and thereby facilitates collusion in that it results in a discount threshold $\underline{\delta}_M$ that is lower than $\underline{\delta}$. However the incentive constraints that impact the strategy set in all punishment periods (they impose the difference

$\pi_i^d(a_{P,s}) - \pi(a_{P,s})$ to be relatively small, all $s \leq l$) do not allow $\underline{\delta}_M$ to attain the lower bound δ^* .

Suppose now that, absent the severity constraint, regime 2 would apply. In that case, recalling that \bar{a}_P is implicitly defined by $\bar{\pi} = \pi_m - \pi_i^d(a_m)$, it is straightforward to observe from the comparison of the expressions of $\bar{\delta}$ and $\underline{\delta}'$, as displayed in Proposition 3 and Lemma 7 respectively, that the two thresholds coincide if and only if $\pi_i^d(\underline{a}_P) = 0$, or equivalently $\underline{a}_P = \bar{a}_P$. When punishments cannot be very severe, in that $\underline{a}_P \succ \bar{a}_P$, firms earn positive profits by deviating from the punishment “floor” (i.e., $\pi_i^d(\underline{a}_P) > 0$, see Assumption 6). In that case there is no finite number of punishment periods that allow firms to implement a_m for a discount level as low as $\bar{\delta}$. That is, $\underline{\delta}_M > \bar{\delta}$. On the other hand, when the most severe punishment is such that firms cannot break even by deviating, so that their minmax profit is negative (i.e., $\pi_i^d(\underline{a}_P) \leq 0$), by lengthening the punishment phase they may implement a_m for any discount level greater than or equal to $\bar{\delta}$, that is $\underline{\delta}_M = \bar{\delta}$.

The next section illustrates the latter results and their interpretation in the context of a linear example.

5 A Linear Example

The example studied in this section emphasizes the importance of considering limited severity constraints. In a standard linear oligopoly structure, we investigate the circumstances which allow firms to sustain perfect collusion (i.e., to maximize joint profits). Toward this aim, we assume that, over all periods, demand is derived from a utility function adapted from Häckner (2000), of the form

$$U(\mathbf{q}, I) = \sum_{i=1}^n q_i - \frac{1}{2} \left(\sum_{i=1}^n q_i^2 + 2\gamma \sum_{i \neq j} q_i q_j \right) + I, \quad (8)$$

which is quadratic in the consumption of q -products and linear in the consumption of the composite I -good (i.e., the numeraire).¹² The parameter $\gamma \in (0, 1)$ measures product substitutability as perceived by consumers. If $\gamma \rightarrow 0$, the demand for the different product varieties are independent and each firm has monopolistic market power, while if $\gamma \rightarrow 1$, the products are perfect substitutes. Consumers maximize utility subject to the budget constraint $\sum p_i q_i + I \leq m$, where

¹²In Häckner (2000), quantities q_i are multiplied by a parameter a_i , that is a measure of the distinctive quality of each variety i . Here we exclude vertical product differentiation by assuming that $a_i = 1$, all $i \in N$.

m denotes income, p_i is the non-negative price of product i , and the price of the composite good is normalized to one. By symmetry, we note $\sum_{j \neq i} q_j = (n-1)q_j$. On the cost side, in the example we set $f = 0$, for simplicity, and $c < 1$. We examine the Cournot version of the model. With quantity-setting firms, the relation q' is more severe than q is formally equivalent to $q' \geq q$.

From (8) firm i 's inverse demand function in each period is

$$p_i(q_i, q_j) = \sup\{0, 1 - q_i - \gamma(n-1)q_j\}, \quad (9)$$

and the inverse demand for each other symmetric firm j in $N \setminus \{i\}$ is

$$p_j(q_i, q_j) = \sup\{0, 1 - \gamma q_i - (1 + \gamma(n-2))q_j\}, \quad (10)$$

all $q_i, q_j \geq 0$, $i \neq j$. It is straightforward to check that a firm's profit function is continuous and the associated maximization problem is convex.

Which of the three regimes we identified in Proposition 1 applies depends on the status of the participation and severity constraints. This in turn depends on the number of firms, the degree of product differentiation, and the marginal (and unit) cost. The connection of the latter cost parameter to the severity constraint, is very intuitive in this example. With linear demand (the quantity demanded is finite at all prices), the most severe punishment is obtained when the price charged by all firms is equal to zero. This may result in exactly zero profits if the marginal cost is equal to zero as well, or to losses if the price-cost margin is negative, all other things (i.e., the demand to each firm) remaining equal. Whether the endogenous q_P^* or \bar{q}_P , as defined above (by simply substituting q for a) is less or more severe than \underline{q}_P can thus be seen to depend only on the comparison of c with a threshold level, we denote by \underline{c} , which is a function of n and γ .

In the specific algebraic context of this example, we may check that (PC) binds if and only if $q_P \leq \tilde{q}_P$, where $\tilde{q}_P = (1-c)/[\gamma(n-1)]$ is computed by solving $\pi^d(q) = 0$ (see Assumption (A5)). Note that, in the absence of fixed costs, we have $\tilde{q}_P = \hat{q}_P$ (see Assumption (A6)). Observe that, because of the absence of fixed costs, deviation profits here cannot be negative (a firm may stop producing to earn zero benefit). Moreover (SC) binds if and only if $q_P \leq \underline{q}_P$, where $\underline{q}_P = 1/[1 + \gamma(n-1)]$ is obtained by solving $p_i(q, q) = 0$. This is because, in the absence of regulatory intervention, the lower bound to punishment profits results from the non-negativity constraint in prices (it binds when quantities are sufficiently large, because demand is finite).

In the appendix we compute a three-part expression of a continuous frontier \underline{c} , which depends on n and γ , so that $\underline{c} = 0$ if $0 \leq \gamma \leq \hat{\gamma}$, $\underline{c} = \underline{c}' > 0$ if $\hat{\gamma} < \gamma \leq \check{\gamma}$, and $\underline{c} = \underline{c}'' > \underline{c}'$ otherwise, with $\hat{\gamma} \equiv 2/(n-1)$ and $\check{\gamma} \equiv 2(1+\sqrt{2})/(n-1)$, all n . This leads to a partition of the parameter space (n, γ, c) into three subsets, one for each regime.

Proposition 6. Given c, n, γ :

1) Regime 1 applies if and only if

- (i) $2 \leq n \leq 3$; $0 \leq \gamma \leq 1$; $0 \leq c < 1$; or
- (ii) $4 \leq n \leq 5$; $0 \leq \gamma \leq 1$; $\underline{c}' \leq c < 1$; or
- (iii) $6 \leq n$; $0 \leq \gamma \leq \hat{\gamma}$; $0 \leq c < 1$; or
- (iv) $6 \leq n$; $\hat{\gamma} \leq \gamma \leq \check{\gamma}$; $\underline{c}' \leq c < 1$.

2) Regime 2 applies if and only if

- $6 \leq n$; $\check{\gamma} \leq \gamma \leq 1$; $\underline{c}'' \leq c < 1$.

3) Regime 3 applies if and only if

- (i) $n = 3$; $\gamma = \hat{\gamma} = 1$; $c = \underline{c} = 0$; or
- (ii) $4 \leq n \leq 5$; $\hat{\gamma} \leq \gamma \leq 1$; $0 \leq c \leq \underline{c}'$; or
- (iii) $6 \leq n$; $\hat{\gamma} \leq \gamma \leq \check{\gamma}$; $0 \leq c \leq \underline{c}'$; or
- (iv) $6 \leq n$; $\check{\gamma} \leq \gamma \leq 1$; $0 \leq c \leq \underline{c}''$.

Proof. See the appendix. ■

If the constant unit cost is sufficiently high (formally, $c \geq \underline{c}$), either regime 1, where neither (PC) nor (SC) binds, or regime 2, where (PC) binds, applies. The former case may hold for all $n \geq 2$, while the second cannot arise if $n < 6$. Regime 3 is ruled out only if $n = 2$. Otherwise, when goods are sufficiently substitutable ($\gamma \geq \hat{\gamma}$), and for all numbers of firms, Proposition 6 indicates that a sufficiently large reduction in c will always result in a shift to regime 3, where (SC) binds. If $n \geq 6$, all three regimes may apply, depending on the values of γ and c . Fig. 3 precisely illustrates this point.¹³

¹³In this figure, $\gamma < (=)\check{\gamma}$ is equivalent to $q_P^* < (=)\tilde{q}_P$ (see the appendix). Hence it is also equivalent to $q_P^* < (=)\bar{q}_P$, from Lemma 4.

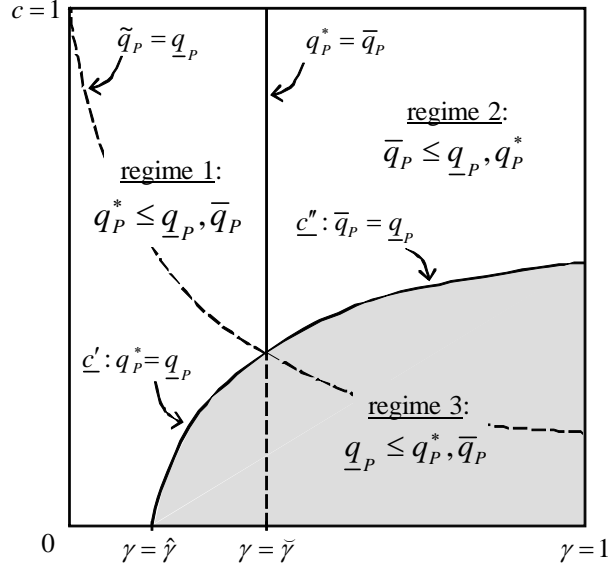


Figure 3: Collusion regimes in plane (c, γ) for $n \geq 6$. The severity constraint binds in the grey area (regime 3).

At any point (n, γ, c) where Regime 3 applies, as represented by the grey area in Fig. 3, the collusive quantity q_m is implementable for all $\delta \geq \underline{\delta}_M$, whenever firms may design an l -period punishment scheme. This illustrates Proposition 4. Moreover, as an illustration of Proposition 5, the grey area can be partitioned into three subsets that describe the consequences of introducing a multi-period punishment scheme. For all points below the frontier \underline{c}'' and above the frontier $\tilde{q}_P = \underline{q}_P$ (so that $\tilde{q}_P \leq \underline{q}_P$ together with $f = 0$ imply $\pi_i^d(\underline{q}_P) = 0$), we have $\underline{\delta}_M = \bar{\delta}$. Then firms may implement a_m for all $\delta \geq \underline{\delta}_M = \bar{\delta}$ with a multi-period punishment. Second, in the grey area below the frontier $\tilde{q}_P = \underline{q}_P$ (in which case $\tilde{q}_P > \underline{q}_P$ implies $\pi_i^d(\underline{q}_P) > 0$) and for $\gamma \geq \tilde{\gamma}$, we have $\underline{\delta}_M > \bar{\delta}$. In that case firms cannot implement a_m for a discount level as low as $\bar{\delta}$. Eventually, for $\gamma < \tilde{\gamma}$ and below \underline{c}' , we have $\underline{q}_P < q_P^* \leq \bar{q}_P$, hence $\delta^* < \underline{\delta}_M < \underline{\delta}$. In words, the severity constraint binds, and several punishments with discounting are only an imperfect substitute for severity in the first period.

The latter figure also helps identifying the role of fixed cost. When $f = 0$, one can check that (IC2) simplifies to the same expression as (PC). This does not hold whenever $f > 0$.¹⁴ In that

¹⁴If $f = 0$ we have $\pi_i^d(q_P) = -f = 0$ for all $q_P \geq \tilde{q}_P = \hat{q}_P$. In that case, the solution to the δ -minimization problem in q_P , under (IC1) and (IC2) only, is the same as the solution under (IC1) and (PC). If $f > 0$ the constraint (PC) becomes stronger than (IC2) for all $q_P \geq \tilde{q}_P$, with $\hat{q}_P > \tilde{q}_P$ (see assumptions (A5) and (A6)).

case all incentive constraints, together with the severity constraint, remain unchanged. The only difference is that the future stream of profits a firm earns from the first period of punishment onward is reduced by the magnitude of fixed costs, so that the participation constraint becomes stronger. Hence the parameter subset where regime 2 applies expands. This has no impact on the discount thresholds δ^* , $\bar{\delta}$, and $\underline{\delta}$ which correspond to each regime. They are displayed as expressions of parameters, together with $\underline{\delta}_M < \underline{\delta}$.

An interesting aspect of Proposition 6 is that the severity constraint can be ignored for all values of c and γ if there are exactly two or three firms (see Regime 1-(i)). In that case, the results obtained in the literature on the implementation of collusion with a duopoly and a constant marginal (and unit) cost normalized to zero are robust to the introduction of a positive constant unit cost, all other specifications remaining the same. This does not apply when $n > 3$.

It is also of interest to compare our result in Proposition 6 with Abreu (1986), where there is no limited severity constraint. In that reference paper, the model is a Cournot oligopoly with a strictly positive constant unit cost, homogenous goods, and a quantity demanded that tends to infinity when the price approaches zero, as specified in $(\tilde{A}1)$ -($\tilde{A}3$) we reproduced above. Then the collusive q_m can be implemented with a two-phase penal code that includes one punishment period only, for all numbers of firms and a given discount factor δ above a threshold. Provided that $\delta \geq \delta^*$, where δ^* is as defined in (2), Proposition 6 extends the latter result to our specific example for any non-negative constant unit cost, and for any degree of product differentiation, if there are at most three firms. This is remarkable since our linear demand specification is *not* a special case of Abreu's class of demand functions. However, with more than three firms, the values of c and/or of γ drive the status of the constraints in the δ -minimization problem. We obtain that c and/or γ must be higher than a threshold for a single-period punishment scheme to implement collusion at $\delta = \delta^*$ or $\delta = \underline{\delta}$.¹⁵

The role of costs, given n and γ , can be illustrated by comparing q_P^* and \bar{q}_P with \underline{q}_P for any c defined on $[0, 1]$. The punishment quantities are represented for all $n \geq 6$ again, with highly substitutable products in Fig. 4(a), where $\tilde{\gamma} < \gamma$, and for more differentiated products in Fig. 4(b), where $\gamma < \tilde{\gamma}$. In both cases regime 3 applies when the constant cost parameter is low, that

We may also assume that $f < 0$ to capture the existence of a profitable outside option. In this case (PC) is weaker than with a non-negative fixed cost.

¹⁵This result contrasts even more sharply with trigger penal code models, in which one can easily check that the sustainability of collusion is not directly connected to the level of marginal costs (at least in the linear cost setup).

is $c \leq \underline{c}$. For higher levels of c we have regime 2 in (a), and regime 1 in (b). The structural boundary level \underline{q}_P depends only on the number of competitors and demand parameters. It is monotone decreasing when either n or γ increases, but constant in c . The optimal punishment quantities q_P^* and \bar{q}_P are linear in the cost parameter and monotone decreasing when it rises closer to 1.

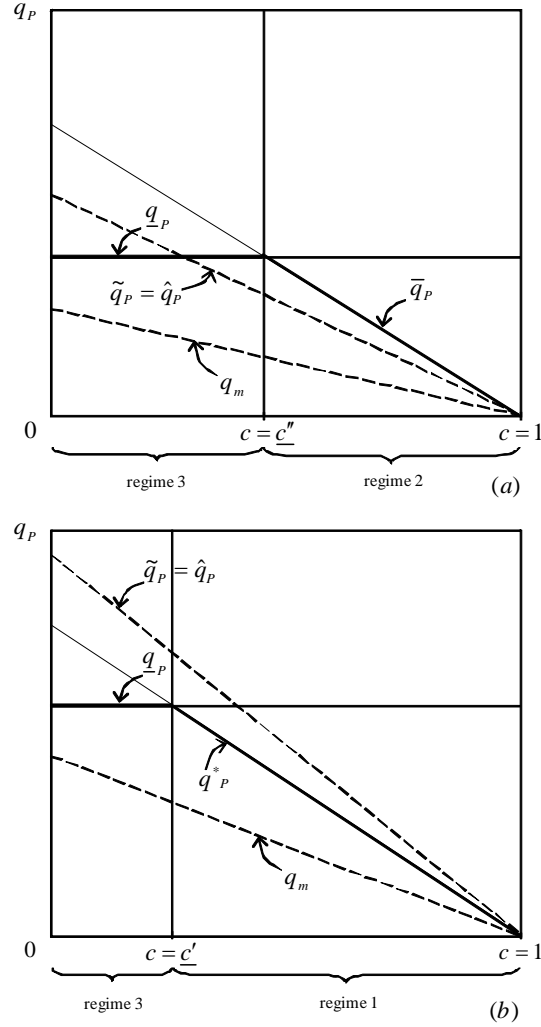


Figure 4: Thick lines represent optimal punishment quantities (all c , and $n \geq 6$). In (a) products are highly substitutable ($\tilde{\gamma} < \gamma$). Regime 3 applies for $c \leq \underline{c}''$, and regime 2 applies otherwise. In (b) products are more differentiated ($\gamma < \tilde{\gamma}$). Regime 3 applies for $c \leq \underline{c}'$, and regime 1 applies otherwise.

We now turn to the impact of a change in the differentiation parameter γ , or in the number of firms n , on the discount factor thresholds δ^* , $\underline{\delta}$, $\bar{\delta}$, and $\underline{\delta}_M$, and also on the cost threshold \underline{c} . From the algebraic expressions displayed in the appendix, it is easy to establish the following monotonicity properties.

Proposition 7. Given n, γ, c :

- (i) \underline{c}' and \underline{c}'' are monotone increasing in n and γ ;
- (ii) δ^* and $\bar{\delta}$ are monotone increasing in n and γ ;
- (iii) $\underline{\delta}$ and $\underline{\delta}_M$ are monotone increasing in n and γ , and monotone decreasing in c .

Proof. It suffices to differentiate the functional forms that appear in the appendix. ■

This proposition establishes that the constant unit cost frontier \underline{c} , together with the lowest values of δ for which the collusive quantity q_m can be implemented, all evolve in the same direction when parameter values change. We find that an increase in product differentiation, and a reduction in the number of firms, facilitate collusion, as they both relax the conditions on c and δ .

6 The Related Literature

A limited severity constraint, as introduced in this paper, plays no role in models *à la* Friedman (1971) where, in case of deviation from the collusive action, trigger strategies call for reversion to the one-shot stage game Nash equilibrium forever (see for example Deneckere (1983, 1984), Majerus (1988), Chang (1991), Ross (1992), Häckner (1994)). In these models (SC) cannot bind since $\underline{a}_P \prec a_{NE}$ by definition. By specifying that firms revert to the Nash equilibrium of the one-shot stage game in all periods subsequent to a deviation, one actually rules out the possibility of modulating the level of punishments in a range where the severity constraint is defined.

A limited severity constraint may impact firms' ability to sustain collusion in more elaborate models that formalize a stick-and-carrot penal code in the spirit of Abreu (1986, 1988). A series of papers in this category investigate the impact of product differentiation and industry concentration on the sustainability of collusive agreements in oligopolistic industries. An example

is Wernerfelt (1989), where more product differentiation may render collusion less sustainable when the number of quantity-setting oligopolists is relatively large.¹⁶ In a repeated Bertrand (i.e., price-setting) model with two firms, spatial horizontal differentiation, and a constant marginal cost set equal to zero, Häckner (1996) demonstrates that there exists an optimal symmetric stick-and-carrot punishment scheme, and confirms that differentiation tends to facilitate collusive agreements. It is also demonstrated that, when the punishment price is constrained to be non-negative, a prolonged price war is an optimal collusive strategy. However in this case there cannot be below cost pricing, a restriction that does not fit most real-world circumstances. Our paper extends the analysis to positive marginal costs, and reveals they *do* impact firms' ability to sustain collusion. With two firms and constant marginal costs again, but with another specification of the horizontal differentiation assumption, Lambertini and Sasaki (2002) find a qualitatively similar relationship between product substitutability and collusion sustainability. This is obtained in a setup where quantities are constrained to be non-negative but prices may fall below zero. The example in the previous section extends this result to a linear setup with n firms, when the severity constraint imposes prices to be non-negative. Lambertini and Sasaki also find that, for all degrees of product differentiation, perfect collusion is less easily sustainable in Bertrand than in Cournot. We confirm this result in the Bertrand version of our linear example of the previous section (with $0 < \gamma < 1$), by finding that, for any degree of product differentiation and all numbers of firms, the frontier \underline{c} and the discount thresholds in all three regimes are higher with price-setting firms than with Cournot players, all other things remaining equal. In other words, for some points in (n, γ, c) regime 3 applies with Bertrand firms, while either regime 1 or 2 occurs with Cournot competition (so that collusion is easier to sustain). The intuition is that incentives to deviate in the Bertrand case are higher than in the Cournot setup, because a deviating firm can capture the whole demand in a price-setting oligopoly. This does not apply in a quantity-setting oligopoly, because a unilateral expansion in some firm's output cannot eliminate its rivals' demand. On the other hand, the severity of punishments does not depend on the nature of competition, but only on structural parameters (that is n , γ , and c in our example). More precisely, the range of per-period punishment profits remains the same in the Bertrand and Cournot cases (they cannot be less than $-cq$).

Another strand of literature investigates the impact of cost asymmetries on collusion. Compte, Jenny, Rey (2002) capitalize on early characterizations by Lambson (1987, 1994) of optimal

¹⁶ Although of interest, this ambiguous result is derived from demand assumptions (adapted from Deneckere, 1983) which are not standard (on this see Osterdal, 2003, pp. 54-55).

punishments – possibly over several periods – for a class of infinitely-repeated games with price-setting sellers of a homogenous good. They examine the impact of the distribution of firm-specific capacity constraints on the ability to sustain collusion. When capacity constraints are weak, in that any subset of firms can serve the entire market, the Nash equilibrium of the stage game also yields zero profit. When aggregate capacity is limited *vis-à-vis* market size, it is shown that asymmetric capacities make collusion more difficult to sustain. With no fixed cost, and a constant marginal cost normalized to zero, firms earn zero profit when they are minmaxed. This holds also when the price is set to zero. Hence the severity constraint associated to price non-negativity can never be binding. Our analysis reveals that another factor would be at play if the marginal cost were specified to be positive. In that case the severity constraint would depend on each firm i 's capacity k_i , with the lowest profit equal to $-ck_i < 0$, and it could be binding. In Vasconcelos (2005), quantity-setting firms have a different share of the industry capital, which determines their marginal costs. In a punishment period, the total industry output is divided in proportion to capital endowments. The analysis focuses on maximum punishments. They make a deviant firm earn its minmax payoff, that is zero (there are no fixed costs), from the first period of punishment onward. In the terms used in the present paper, this is equivalent to assuming that firms' punishment quantities are such that the participation constraint binds. When this holds, an important result is that a two-phase stick-and-carrot penal code exists, where the collusive action leads to monopoly profits (perfect collusion), if the discount factor is higher than a threshold level that depends on the size of the largest firm. The introduction of our severity constraint – which is a natural extension since demand is finite so that punishments are structurally limited from below – would lead to a higher threshold for some parameter values. By choosing simple values for the cost and demand parameters in Vasconcelos (2005), we find that the above mentioned discount threshold remains unchanged only if the marginal cost parameter is sufficiently high. More specifically, by setting (say) $k_i = 1/n$ for each firm i 's capital share (so that symmetry is restored), and $a = b = 1$ for the linear demand curve parameters, one obtains that $\bar{q}_P < \underline{q}_P$ if and only if $c > \sup \{0, \bar{c}(\delta)\}$, where \bar{q}_P is the quantity such that both (IC1) and (PC), as defined in the present paper, are exactly satisfied, and \underline{q}_P is the quantity that drives prices to zero.¹⁷

¹⁷With $k_i = 1/n$ and $a = b = 1$, the discount threshold of Proposition 2 in Vasconcelos (2005, p. 48) reduces to $3(n+2)n/(2n+1)^2$. With δ at the latter level, we obtain $\bar{c}(\delta) = 1/n$.

7 Appendix

7.1 Single-Period Punishments

Lemma 1. *The optimal single-period punishment action a_P^* and the discount factor lower bound δ^* are such that (IC1) and (IC2) hold with equality.*

Proof. Suppose that $a = a_P^*$, the optimal punishment, and $\delta = \delta^*$, the lowest possible discount factor for which a_m is implementable. There are three possible cases: either the two inequalities are slack, or only one, or none. Consider the first two cases in turn. (i) If none of the two constraints binds, observe that the two expressions on the right-hand side of the inequality sign are continuous in δ and monotonically decreasing when the discount parameter is decreasing, so that there exists $\delta' < \delta^*$ such that the system still holds true when $\delta = \delta'$, contradicting the claim that δ^* is a lower bound. (ii) If exactly one constraint binds for $\delta = \delta' < \delta^*$, recall that profit functions $\pi_i^d(\cdot)$ and $\pi(\cdot)$ are continuous in firms' choices, therefore by changing slightly the punishment action from a_P^* to a_P' one can relax the binding constraint and still let the other inequality be verified. This leads the two constraints (IC1) and (IC2) to be slack, implying again that there exists $\delta'' < \delta'$ such that the system still holds true when $\delta = \delta''$. It follows from (i) and (ii) that both constraints must be binding. ■

Proposition 1. *The collusive action a_m is implementable with a single-period punishment if and only if $\delta \geq \delta_1^*$, with*

$$\delta_1^* = \begin{cases} \delta^* \equiv \frac{\pi_i^d(a_m) - \pi_m}{\pi_m - \pi(a_P^*)} & \text{if } a_P^* \succeq \underline{a}_P, \bar{a}_P \quad (\text{regime 1}); \\ \bar{\delta} \equiv \frac{\pi_i^d(a_m) - \pi_m}{\pi_m - \pi} & \text{if } \bar{a}_P \succeq \underline{a}_P, a_P^* \quad (\text{regime 2}); \\ \underline{\delta} \equiv \frac{\pi_i^d(a_m) - \pi_m}{\pi_m - \underline{\pi}} & \text{if } \underline{a}_P \succeq a_P^*, \bar{a}_P \quad (\text{regime 3}). \end{cases}$$

Proof. There are three steps. First we solve a less constrained version of (1), in which (PC) and (SC) are absent. Then we reintroduce each of the latter two constraints separately.

1) Consider the δ -minimization problem without constraints (PC) and (SC). The two constraints (IC1-IC2) can be rewritten together as

$$X(\delta) \leq \pi_m - \pi(a_P) \leq Y(\delta, a_P), \quad (11)$$

where $X(\delta) \equiv [\pi_i^d(a_m) - \pi_m] / \delta$ and $Y(\delta, a_P) \equiv [\pi_m - \pi_i^d(a_P)] / (1 - \delta)$ denote the lower-bound and the upper-bound, respectively, of the profit differential $\pi_m - \pi(a_P)$. (They are represented in Fig. 2.) We know that (a_P^*, δ^*) solves $X(\delta) = Y(\delta, a_P)$ from Lemma 1.¹⁸ Together with $\pi_i^d(a_m) - \pi_i^d(a_P^*) = \pi_m - \pi(a_P^*)$ from Lemma 2, this leads to

$$\delta^* = \frac{\pi_i^d(a_m) - \pi_m}{\pi_m - \pi(a_P^*)}. \quad (12)$$

2) Introduce (PC) in addition to $(IC1-IC2)$. For $a_P = a_P^*$ recall that the latter two constraints imply $X(\delta) \leq \pi_m - \pi(a_P^*) \leq Y(\delta, a_P^*)$, while PC can be rewritten $\pi_m - \pi(a_P^*) \leq \bar{Y}(\delta)$, with $\bar{Y}(\delta) \equiv \pi_m / (1 - \delta)$. There are two cases:

(i) If $\bar{a}_P \prec a_P^*$ then $\tilde{a}_P \prec a_P^*$, from Lemma 4. Then we know from (PC) that $\bar{Y}(\delta) > Y(\delta, a_P^*)$ for all $\delta \in [0, 1)$, and the participation constraint is slack for $a_P = a_P^*$ and $\delta = \delta^*$.

(ii) If $a_P^* \preceq \bar{a}_P$ then $a_P^* \preceq \tilde{a}_P$, from Lemma 4. Then we know from (PC) that $\bar{Y}(\delta) \leq Y(\delta, a_P^*)$ for all $\delta \in [0, 1)$. When the inequality sign is strict (PC) is violated for $a_P = a_P^*$ and $\delta = \delta^*$. Next, toward a participation-constrained solution, substitute (PC) for $(IC2)$, or equivalently $\bar{Y}(\delta)$ for $Y(\delta, a_P)$ in (11). (See Fig. 2.) The negative slope of $X(\delta)$, together with the monotonicity of $\pi(a_P)$, imply that the minimizer \bar{a}_P and the minimum $\bar{\delta}$ verify $X(\bar{\delta}) = \pi_m - \bar{\pi} = \bar{Y}(\bar{\delta})$.¹⁹ This leads to

$$\bar{\delta} = \frac{\pi_i^d(a_m) - \pi_m}{\pi_m - \bar{\pi}}, \quad (13)$$

and then one checks that $\bar{Y}(\bar{\delta}) \leq Y(\bar{\delta}, \bar{a}_P)$.

(iii) Clearly if $\bar{a}_P \succ (=) \underline{a}_P$, then any (δ, \bar{a}_P) , with $\delta \geq \bar{\delta}$, also verifies (SC) .

3) Introduce (SC) , in addition to $(IC1-IC2)$. Then for $a_P = a_P^*$, the severity constraint can be rewritten $\pi_m - \pi(a_P^*) \leq \underline{Y}$, where $\underline{Y} \equiv \pi_m - \underline{\pi}$. There are two cases:

(i) If $\underline{a}_P \prec a_P^*$ we have $\underline{\pi} < \pi(a_P^*)$, hence (SC) is slack for $a_P = a_P^*$, all δ .

¹⁸ Deviation profits $\pi_i^d(a_P)$ have a lower bound (a firm may always stop selling; see (A6)), all a_P . Therefore $\lim_{\delta \rightarrow 0} X(\delta) = +\infty > Y(0, a_P) = \pi_m - \pi_i^d(a_P)$, and $X(1) = \pi_i^d(a_m) - \pi_m < \lim_{\delta \rightarrow 1} Y(\delta, a_P) = +\infty$. Hence there always exists $\delta^*(a_P)$ in $[0, 1)$ verifying $X(\delta^*(a_P)) = Y(\delta^*(a_P), a_P)$, all a_P .

¹⁹ Note that $\lim_{\delta \rightarrow 0} X(\delta) = +\infty > \bar{Y}(0) = \pi_m$ together with $\lim_{\delta \rightarrow 1} X(\delta) = \pi_i^d(a_m) - \pi_m < \lim_{\delta \rightarrow 1} \bar{Y}(\delta) = +\infty$ imply that there always exists $\bar{\delta}$ in $[0, 1)$ verifying $X(\bar{\delta}) = \bar{Y}(\bar{\delta})$.

(ii) If $a_P^* \preceq \underline{a}_P$, we know from (SC) that $\underline{Y} \leq X(\delta^*) = \pi_m - \pi(a_P^*) = Y(\delta^*, a_P^*)$. When the inequality sign is strict (SC) is violated for $a_P = a_P^*$ and $\delta = \delta^*$. Next, toward a severity-constrained solution, one substitutes (SC) for (IC2), or equivalently \underline{Y} for $Y(\delta, a_P)$ in (11). (See Fig. 2.) Since \underline{Y} is a constant and $\pi_m - \pi(a_P)$ reaches a maximum for $a_P = \underline{a}_P$, the minimum $\underline{\delta}$ verifies $X(\underline{\delta}) = \underline{Y}$.²⁰ This leads to

$$\underline{\delta} = \frac{\pi_i^d(a_m) - \pi_m}{\pi_m - \underline{\pi}}. \quad (14)$$

Then $a_P^* \preceq \underline{a}_P \preceq a_m$ together with assumption (A7) imply that $\underline{\delta} \geq \frac{\pi_i^d(\underline{a}_P) - \underline{\pi}}{\pi_m - \underline{\pi}}$, hence that $\underline{Y} \leq Y(\underline{\delta}, \underline{a}_P)$.

(iii) Clearly if $\underline{a}_P \succ (=) \bar{a}_P$, then any $(\delta, \underline{a}_P)$, with $\delta \geq \underline{\delta}$, also verifies (PC). ■

7.2 Multi-Period Punishments

Lemma 5. *Given $a_{P,1}$, the lowest discount factor δ verifying (MIC 1-MIC 2) results from punishment actions $a_{P,k} \preceq a_m$, with $k > 1$ (where at least one punishment action is strictly more severe than a_m), such that the two multi-period incentive constraints (MIC 1) and (MIC 2) bind.*

Proof. In (MIC 1), the expression on the right-hand side of the weak inequality sign simplifies to $\sum_{k=1}^{l+1} \delta^k [\pi_m - \pi(a_{P,k})]$. It is clearly monotone increasing when either $a_{P,k}$ decreases, all $k \geq 1$, or when δ increases, the left-hand side expression (which does not depend on punishment levels) remaining constant. In (MIC 2), the expression on the right-hand side of the weak inequality sign can be rewritten $\delta [(1 - \delta) V_1(\mathbf{a}_P, \delta) - \pi(a_{P,1})]$. It is monotone increasing when $a_{P,k}$ increases (since $\delta(1 - \delta) > 0$), for all $k > 1$, the left-hand side expression (a function of $a_{P,1}$ only) remaining constant. Then for any given $a_{P,1}$, suppose that $a_{P,2}, \dots, a_{P,l}$ are such that δ takes the lowest possible value for which (MIC 1-MIC 2) hold true. There are three possible cases: either the two inequalities are slack, or only one, or none. (i) If none of the two constraints binds, by continuity, one may obviously reduce δ by an arbitrarily small amount so that both constraints remain verified, contradicting the claim that there is no lower discount factor verifying (MIC 1) and (MIC 2). (ii) If exactly one of the two constraints binds, pick any $k > 1$ such that $a_{P,k} \prec a_m$. Then by continuity, one may reduce δ and adjust $a_{P,k}$ so

²⁰Since $X(\delta)$ is downward sloping, and $\lim_{\delta \rightarrow 0} X(\delta) = +\infty > \underline{X}$, there exists $\underline{\delta}$ in $[0, 1)$ verifying $X(\underline{\delta}) = \underline{Y}$ if and only if $\lim_{\delta \rightarrow 1} X(\delta) < \underline{Y}$. We focus here on situations in which this holds. Otherwise a_m is not implementable.

that the right-hand side expression of the binding constraint remains constant, while the other constraint remains satisfied, contradicting again the initial supposition. Therefore it must be the case that, given $a_{P,1}$, (*MIC 1-MIC 2*) hold with an equality sign when $a_{P,2}, \dots, a_{P,l}$ are such that δ is minimized. ■

Proposition 2. *In the multi-period punishment scheme, if $a_P^* \succeq \bar{a}_P, \underline{a}_P$ the collusive action a_m is implementable if and only if $\delta \geq \delta^*$, and $\mathbf{a}_P^* \equiv (a_P^*, a_m, \dots, a_m)$ is optimal.*

Proof. There are two steps: (1) We investigate a less constrained version of (7) by leaving aside the last $l - 1$ multi-period incentive constraints together with (*MPC*) and (*MSC*), to keep only (*MIC 1*) and (*MIC 2*). This is done by capitalizing on Lemma 5: we solve in (δ, V_1) the system (*MIC 1-MIC 2*) with equality signs, to obtain $(\delta^*(a_{P,1}), V_1(\mathbf{a}_P, \delta^*(a_{P,1})))$; then we identify the level of $a_{P,1}$ that minimizes $\delta^*(a_{P,1})$ under the feasibility constraint that $V_1(\mathbf{a}_P, \delta^*(a_{P,1})) \leq V_l(\mathbf{a}_P, \delta^*(a_{P,1})) = \pi_m / (1 - \delta^*(a_{P,1}))$. This leads to the minimizer $a_{P,1}^* = a_P^*$. (2) We show that $(\delta^*(a_P^*), V_1(\mathbf{a}_P, \delta^*(a_P^*)))$ satisfies all incentive constraints in (*MIC 1-MIC l+1*) as well as (*MPC-MSC*).

(1) Consider the δ -minimization problem with the two incentive constraints (*MIC 1*) and (*MIC 2*) only. Observing that $V_l(\mathbf{a}_P, \delta) = V_0(\mathbf{a}_m, \delta)$, the two constraints become

$$X(\delta) \leq V_0(\mathbf{a}_m, \delta) - V_0(\mathbf{a}_P, \delta) \leq Y(\delta, a_{P,1}), \quad (15)$$

where $X(\delta) \equiv [\pi_i^d(a_m) - \pi_m] / \delta$ and $Y(\delta, a_{P,1}) \equiv [\pi_m - \pi_i^d(a_{P,1})] / (1 - \delta)$ denote the lower-bound and the upper-bound, respectively, of the value differential $V_0(\mathbf{a}_m, \delta) - V_0(\mathbf{a}_P, \delta) = V_0(\mathbf{a}_m, \delta) - \pi(a_{P,1}) - \delta V_1(\mathbf{a}_P, \delta)$. Given $a_{P,1}$, from Lemma 5 we know that (15) must hold with an equality sign throughout for δ to be minimized. Solving $X(\delta) = Y(\delta, a_P)$ in $(\delta, V_1(\mathbf{a}_P, \delta))$, we find

$$\delta^*(a_{P,1}) = \frac{\pi_i^d(a_m) - \pi_m}{\pi_i^d(a_m) - \pi_i^d(a_{P,1})}, \quad (16)$$

and

$$V_1(\mathbf{a}_P, \delta^*(a_{P,1})) = [\pi_i^d(a_m) - \pi_i^d(a_{P,1}^*)] \left(\frac{\pi_i^d(a_{P,1}^*) - \pi(a_{P,1}^*)}{\pi_i^d(a_m) - \pi_m} + \frac{\pi_i^d(a_{P,1}^*)}{\pi_m - \pi_i^d(a_{P,1}^*)} \right). \quad (17)$$

Observe from the monotonicity of $\pi_i^d(a_{P,1})$ in $a_{P,1}$ (Lemma 3) that $\delta^*(a_{P,1})$ is monotone non-decreasing in $a_{P,1}$. Therefore the lowest value of $\delta^*(a_{P,1})$ is obtained for the most severe first-period punishment $a_{P,1}$ compatible with the feasibility constraints of the problem. Note in

particular from (5) that $a_{P,1}$ must be such that $V_s(\mathbf{a}_P, \delta) \leq V_t(\mathbf{a}_P, \delta) \leq V_l(\mathbf{a}_P, \delta) = \pi_m / (1 - \delta)$, all $s \leq t \leq l$. Then $V_1(\mathbf{a}_P, \delta) \leq \pi_m / (1 - \delta)$, together with (16) and (17), becomes

$$\left[\pi_m - \pi_i^d(a_{P,1}) \right] \left(1 - \frac{\pi_i^d(a_{P,1}) - \pi(a_{P,1})}{\pi_i^d(a_m) - \pi_m} \right) \geq 0. \quad (18)$$

Clearly $\pi_m - \pi_i^d(a_{P,1})$ for all $a_{P,1} \preceq a_{NE}$ (since the monotonicity of $\pi_i^d(a_P)$ implies that $\pi_i^d(a_{P,1}) \leq \pi_i^d(a_{NE}) = \pi(a_{NE})$, while $\pi(a_{NE}) < \pi_m$ for all $a_{NE} \prec a_m$). It follows from (18) that the term between rounded brackets must be non-negative. This implies that

$$\pi_i^d(a_{P,1}) - \pi(a_{P,1}) \leq \pi_i^d(a_m) - \pi_m. \quad (19)$$

Recalling from Lemma 2 that $\pi_i^d(a_P^*) - \pi(a_P^*) = \pi_i^d(a_m) - \pi_m$, from Assumption (A7) we obtain that $a_{P,1}$ cannot be strictly more severe than a_P^* .

(2) Substitute a_P^* for $a_{P,1}$ in (16 – 17), and also $\pi_i^d(a_m) - \pi_m$ for $\pi_i^d(a_P^*) - \pi(a_P^*)$, again from Lemma 2, to obtain

$$\delta^*(a_P^*) = \delta^* \equiv \frac{\pi_i^d(a_m) - \pi_m}{\pi_i^d(a_m) - \pi_i^d(a_P^*)},$$

and

$$V_1^*(a_P^*, \delta^*(a_P^*)) = \frac{\pi_m}{1 - \delta^*}.$$

It follows directly from the later equation that $V_1^*(a_P^*, \delta^*(a_P^*)) = V_l(\mathbf{a}_P, \delta^*(a_P^*))$, implying that $\pi(a_{P,k}^*) = \pi_m$, all $k > 1$. This says that $\mathbf{a}_P^* = (a_P^*, a_m, \dots, a_m)$ when the only the two incentive constraints in (MIC 1) and (MIC 2 are considered. Next, observe from the definition of continuation profits in (5) that $a_{P,k}^* = a_m$, all $k > 1$, implies that $V_1(\mathbf{a}_P^*, \delta) = V_s(\mathbf{a}_P^*, \delta)$, all s . It follows that the last $l - 1$ multi-period incentive constraints are all identical to the first one, that is (MIC 1), implying that all constraints in (MIC 1-MIC l+1) are satisfied. Since $a_P^* \succeq \bar{a}_P, \underline{a}_P$ it is also plain that (MPC) and (MSC) are satisfied. Therefore the solution to the less constrained problem is also a solution to (7), and the punishment (a_P^*, a_m, \dots, a_m) is optimal. ■

Proposition 3. *In the multi-period punishment scheme, if $\bar{a}_P \succeq \underline{a}_P, a_P^*$, the collusive action a_m is implementable if and only if $\delta \geq \bar{\delta} \equiv \frac{\pi_i^d(a_m) - \pi_m}{\pi_m - \bar{\pi}}$, and $\bar{\mathbf{a}}_P \equiv (\bar{a}_P, a_m, \dots, a_m)$ is optimal.*

Proof. There are two steps: (1) In addition to (MIC 1) and (MIC 2), we introduce (MPC) in the less constrained version of (7), the last $l - 1$ multi-period incentive constraints and (MSC) being

left aside. We show that (MPC) is stronger than $(IC2)$ if $a_P^* \preceq \tilde{a}_P$. Then a_m is implementable with the l -period punishment $\bar{\mathbf{a}}_P \equiv (\bar{a}_P, a_m, \dots, a_m)$ if $\delta = \bar{\delta}$, that is the lower bound to the interval of δ for which $(MIC\ 1)$ and (MPC) are compatible. (2) We obtain that $(\bar{\delta}, \bar{\mathbf{a}}_P)$ satisfies all other incentive constraints $(MIC3-MICl+1)$, in which case $\bar{\delta}$ is a solution of (7) and $\bar{\mathbf{a}}_P$ is optimal.

(1) Introduce the multi-period participation constraint (MPC) in addition to $(MIC\ 1-MICl+1)$. For $\mathbf{a}_P = \mathbf{a}_P^* \equiv (a_P^*, a_m, \dots, a_m)$ recall that the first two incentive constraints in $(MIC\ 1)$ and $(MIC\ 2)$ can be rewritten $X(\delta) \leq V_0(\mathbf{a}_m, \delta) - V_0(\mathbf{a}_P^*, \delta) \leq Y(\delta, \mathbf{a}_P^*)$, while (MPC) can be rewritten $V_0(\mathbf{a}_m, \delta) - V_0(\mathbf{a}_P^*, \delta) \leq \bar{Y}(\delta)$, with $\bar{Y}(\delta) \equiv \pi_m / (1 - \delta)$. If $\bar{a}_P \succeq a_P^*$ we know from Lemma 4 that $\tilde{a}_P \succeq a_P^*$, in which case $\pi_i^d(a_P^*) \leq 0$ from (A5). This implies that $\bar{Y}(\delta) \leq Y(\delta, a_P^*)$ for any $\delta \in [0, 1)$. When the inequality sign is strict (MPC) is stronger than $(MIC\ 2)$, and thus is violated for $\mathbf{a}_P = \mathbf{a}_P^*$ and $\delta = \delta^*$. Next, toward a participation-constrained solution, substitute (MPC) for $(MIC\ 2)$. From Proposition 1, in the single-period punishment case we know that $(IC1)$ and (PC) are satisfied if $a_P = \bar{a}_P$ and $\delta \geq \bar{\delta}$, implying that in the multi-period setup $(MIC\ 1)$ and (MPC) are satisfied as well if $\bar{\mathbf{a}}_P \equiv (\bar{a}_P, a_m, \dots, a_m)$ and $\delta \geq \bar{\delta}$. This is sufficient to conclude that there is at least one punishment \mathbf{a}_P for which a_m is implementable with $\delta \geq \bar{\delta}$. Then recall from Lemma 6 that $\bar{\delta}$ is the lowest value of δ compatible with $(MIC\ 1)$ and (MPC) . This is sufficient to conclude that $\bar{\delta}$ is a solution to the δ -minimization problem under the constraints $(MIC\ 1)$, $(MIC\ 2)$, (MPC) .

(2) Observe from the definition of continuation profits in (5) that $a_{P,k} = a_m$ for all $k > 1$ implies that $V_s(\mathbf{a}_P, \delta) = V_0(\mathbf{a}_m, \delta)$, all $s > 1$. It follows the last $l - 1$ multi-period incentive constraints are all identical to $(MIC\ 1)$, implying that all constraints in $(MIC\ 1-MICl+1)$ are satisfied. Clearly if $\bar{a}_P \succ (=) \underline{a}_P$, then $(\bar{\delta}, \bar{\mathbf{a}}_P)$ also verifies (MSC) . Therefore $\bar{\delta}$ is a solution to (7), and the punishment $(\bar{a}_P, a_m, \dots, a_m)$ is optimal, all l . ■

Lemma 8. For all \underline{V} verifying $\underline{\pi} < (1 - \delta)\underline{V} \leq \pi_m$, there exists a finite l and a punishment $\underline{\mathbf{a}}_P \equiv (\underline{a}_P, a_{P,2}, \dots, a_{P,k}, \dots, a_{P,l})$, with $a_{P,k} \succeq \underline{a}_P$ for all $k > 1$, such that $V_1(\underline{\mathbf{a}}_P, \delta) = \underline{V}$.

Proof. There are three steps: (1) we show that, given any δ , for any $l \geq 2$ there exists a punishment $\underline{\mathbf{a}}_P^l$ of length l such that $V_1(\underline{\mathbf{a}}_P^l, \delta) = \underline{V}$ for any \underline{V} in a closed interval I_l we define; (2) we establish that the upper-bound of I_{l+1} is the lower bound of I_l so that their finite union

$I_L = \cup_{l=1}^L I_l$ is itself a closed interval; (3) we conclude by evidencing that the lower and upper bounds of the union of intervals are respectively $\underline{\pi}/(1-\delta)$ and $\pi_m/(1-\delta)$.

(1) Define $\mathbf{a}_P^l \equiv (a_{P,1}^l, a_{P,2}^l, \dots, a_{P,k}^l, \dots, a_{P,l}^l)$, where $a_{P,k}^l = \underline{a}_P$ for all $k = 1, 2, \dots, l-1$, and $a_{P,l}^l \succeq \underline{a}_P$. Here firms opt for the most severe action \underline{a}_P in the first $l-1$ periods, and for a possibly softer action in the l -th period. In the latter final period, the continuity of π in a_P^l implies that $\pi(a_P^l)$ may take any value in $[\pi(\underline{a}_P), \pi_m]$. Let $\underline{\mathbf{a}}_P^l$ and $\bar{\mathbf{a}}_P^l$ denote the just defined penal code \mathbf{a}_P^l where $a_{P,l}^l = \underline{a}_P$ and $a_{P,l}^l = a_m$ respectively. By definition, for any value \underline{V} in $I_l = [V_1(\underline{\mathbf{a}}_P^l, \delta), V_1(\bar{\mathbf{a}}_P^l, \delta)]$, there exists \mathbf{a}_P^l such that $V_1(\mathbf{a}_P^l, \delta) = \underline{V}$.

(2) Clearly, $V_1(\underline{\mathbf{a}}_P^l, \delta) = V_1(\bar{\mathbf{a}}_P^{l+1}, \delta)$ so that $I_L = \cup_{l=1}^L I_l = [V_1(\underline{\mathbf{a}}_P^L, \delta), V_1(\bar{\mathbf{a}}_P^1, \delta)]$ for any integer $L > 1$, and

(3) from the definition of continuation profits in (5) we know that $V_1(\bar{\mathbf{a}}_P^1, \delta) = \pi_m/(1-\delta)$, while $V_1(\underline{\mathbf{a}}_P^L, \delta)$ verifies

$$(1-\delta)V_1(\underline{\mathbf{a}}_P^L, \delta) = \underline{\pi} + \delta^{L-1}(\pi_m - \underline{\pi}).$$

Since $\lim_{L \rightarrow \infty} (\underline{\pi} + \delta^{L-1}(\pi_m - \underline{\pi})) = \underline{\pi}$, for any $\underline{V} > \underline{\pi}/(1-\delta)$ there exists a finite L such that $\underline{\pi} + \delta^{L-1}(\pi_m - \underline{\pi}) \leq (1-\delta)\underline{V}$ so that $\underline{V} \in [V_1(\underline{\mathbf{a}}_P^L, \delta), V_1(\bar{\mathbf{a}}_P^1, \delta)]$, and there exists a punishment profile \mathbf{a}_P^l , with $l \leq L$, such that $V_1(\mathbf{a}_P^l, \delta) = \underline{V}$. ■

Remark 3. If $\bar{a}_P \succ a_P^*$ there is an infinite number of punishments that permit firms to implement a_m for $\delta = \bar{\delta}$.

Proof. Recall from proof of Proposition 3 that (MIC 1) is written as $X(\delta) \leq V_0(\mathbf{a}_m, \delta) - V_0(\mathbf{a}_P, \delta)$, and (MPC) as $V_0(\mathbf{a}_m, \delta) - V_0(\mathbf{a}_P, \delta) \leq \bar{Y}(\delta)$, with $X(\delta) \equiv [\pi_i^d(a_m) - \pi_m]/\delta$ and $\bar{Y}(\delta) \equiv \pi_m/(1-\delta)$. If $(\mathbf{a}_P, \delta) = (\bar{\mathbf{a}}_P, \bar{\delta})$ and $\bar{a}_P \succ a_P^*$ we know that $X(\bar{\delta}) = V_0(\mathbf{a}_m, \bar{\delta}) - V_0(\bar{\mathbf{a}}_P, \bar{\delta}) = \bar{Y}(\delta)$, where $\bar{\mathbf{a}}_P \equiv (\bar{a}_P, a_m, \dots, a_m)$, while all other multi-period incentive constraints are satisfied also. Given $\bar{\delta}$, consider a change from $\bar{\mathbf{a}}_P$ to $\bar{\mathbf{a}}_P'$, with $a_{P,1}' \succ \bar{a}_P$ and $a_{P,k}' \preceq a_m$ for some $k > 1$, that verifies $V_0(\bar{\mathbf{a}}_P, \delta) - V_0(\bar{\mathbf{a}}_P', \delta) = 0$. For all $l > 1$, the continuity of $\pi(a_{P,k})$ in $a_{P,k}$ implies that the number of solutions $\bar{\mathbf{a}}_P'$ to the latter equation is infinite. By the very nature of the change both constraints (MIC 1) and (MPC) remain exactly satisfied, while by continuity (MIC 2) remains satisfied as well for a sufficiently small adjustment (it was slack

for $a_{P,1} = \bar{a}_P$). Moreover, the $l - 1$ remaining multi-period incentive constraints in $(MIC3 - MICl + 1)$ are relaxed as a result of an adjustment from a_m “down” to $a'_{P,k} \prec a_m$ in any of the $k > 1$ following periods of punishment, all other things remaining equal. It follows that a_m is implementable if $(\mathbf{a}_P, \delta) = (\bar{\mathbf{a}}'_P, \bar{\delta})$. ■

Proposition 4. *In the multi-period punishment scheme, if $\underline{a}_P \succeq a_P^*, \bar{a}_P$, there exists a finite $l > 1$ such that the collusive action a_m is implementable if and only if $\delta \geq \sup\{\underline{\delta}', \bar{\delta}\} \equiv \underline{\delta}_M$.*

Proof. There are two cases that depend on the comparison of $\underline{\delta}'$ and $\bar{\delta}$. In both cases: (1) we establish that there exists a finite punishment, we denote \mathbf{a}_P , which is such that $V_1(\mathbf{a}_P, \delta)$ is equal to a particular value we explicit; (2) we check that all incentive constraints are satisfied; (3) we also verify that the participation and severity constraints hold.

$$(\bar{\delta} \leq \underline{\delta}')$$

(1) Define implicitly $\underline{\mathbf{a}}_P$, specified to take the form of \mathbf{a}_P^l as introduced in Lemma 8 (so that (MSC) is satisfied) by

$$V_1(\underline{\mathbf{a}}_P, \delta) = \frac{1}{1 - \delta} \left(\underline{\pi} + \frac{\pi_i^d(\underline{a}_P) - \underline{\pi}}{\delta} \right) \quad (20)$$

which describes continuation profits from the 2nd period of punishment onward. Given δ , from Lemma 8, a sufficient condition for $\underline{\mathbf{a}}_P$ to be well defined is $\underline{\pi} < (1 - \delta) V_1(\underline{\mathbf{a}}_P, \delta) \leq \pi_m$. To check this holds, consider the two inequalities in turn: (i) We have $\underline{\pi} < (1 - \delta) V_1(\underline{\mathbf{a}}_P, \delta)$ since $[\pi_i^d(\underline{a}_P) - \underline{\pi}] / \delta > 0$ (by definition). (ii) Toward $V_1(\underline{\mathbf{a}}_P, \delta) \leq \pi_m / (1 - \delta)$ first note that $\underline{a}_P \succeq a_P^*$ implies that $\pi_i^d(a_m) - \pi_m \geq \pi_i^d(\underline{a}_P) - \pi(\underline{a}_P)$ from Assumption (A7) and Lemma 2. From the expression of $\underline{\delta}'$, as displayed in Lemma 7, it follows that

$$\underline{\delta}' \leq \frac{\pi_i^d(a_m) - \pi_m}{\pi_m - \pi(\underline{a}_P)}. \quad (21)$$

Then pick $\delta = \underline{\delta}'$. Now (21), and $X(\underline{\delta}') \equiv [\pi_i^d(a_m) - \pi_m] / \underline{\delta}' = V_0(\mathbf{a}_m, \underline{\delta}') - V_0(\underline{\mathbf{a}}_P, \underline{\delta}')$, imply that $\pi_m - \pi(\underline{a}_P) \leq V_0(\mathbf{a}_m, \underline{\delta}') - V_0(\underline{\mathbf{a}}_P, \underline{\delta}')$. Moreover, substituting $(1 - \underline{\delta}') V_0(\mathbf{a}_m, \underline{\delta}')$ for π_m in the latter expression leads to $V_0(\underline{\mathbf{a}}_P, \underline{\delta}') \leq \underline{\delta}' V_0(\mathbf{a}_m, \underline{\delta}') + \pi(\underline{a}_P)$. Then substituting $\pi(\underline{a}_P) + \underline{\delta}' V_1(\underline{\mathbf{a}}_P, \underline{\delta}')$ for $V_0(\underline{\mathbf{a}}_P, \underline{\delta}')$ results in $(1 - \underline{\delta}') V_1(\underline{\mathbf{a}}_P, \underline{\delta}') \leq \pi_m$, as needed. As $(1 - \delta) V_1(\underline{\mathbf{a}}_P, \delta)$ is monotone decreasing in δ , it follows that $(1 - \delta) V_1(\underline{\mathbf{a}}_P, \delta) \leq \pi_m$ for all $\delta \geq \underline{\delta}'$. Eventually, (i) and (ii) establish that there exists at least one $\underline{\mathbf{a}}_P$ for a finite l such that $V_1(\underline{\mathbf{a}}_P, \delta)$ satisfies (20) for all $\delta \geq \underline{\delta}'$.

(2) At $\delta = \underline{\delta}'$, we have that $(MIC\ 1-MIC\ 2)$ are exactly satisfied for $\mathbf{a}_P = \underline{\mathbf{a}}_P$ (which is as defined in Lemma 8 so that $a_{P,1} = \underline{a}_P$). Clearly, $V_{k+1}(\underline{\mathbf{a}}_P, \delta)$ is strictly increasing in k as long as $k \leq l-1$. Since $\pi_i^d(a_{P,k}) - \pi(a_{P,k})$ is identical for all $1 \leq k \leq l-1$, if $(MIC\ 2)$ holds and is binding, it must be the case that all constraints $(MIC3), \dots, (MICl)$ hold also and are slack. There is no loss of generality in assuming that $a_{P,l} \prec a_m$. (If there is equality, collusion can be implemented by the means of a $l-1$ punishment scheme where $a_{P,l-1} = \underline{a}_P \prec a_m$). Assuming this is the case, we know from Assumption (A7) and Lemma 2 that $\pi_i^d(a_{P,l}) - \pi(a_{P,l}) < \pi_i^d(a_m) - \pi_m$. If $(MIC\ 1)$ holds and is binding, it must be the case that $(MIC\ 1+1)$ holds also and is slack. This says that, in the absence of participation constraint, a_m is implementable with at least one l -punishment vector, that is $\mathbf{a}_P = \underline{\mathbf{a}}_P$, when $\delta = \underline{\delta}'$. Since for $\mathbf{a}_P = \underline{\mathbf{a}}_P$ all MICs $(MIC\ 1-MIC\ 1+1)$ are monotone increasing in δ , this also holds for all $\delta \geq \underline{\delta}'$.

(3) Consider now the participation constraint. If $\bar{\delta} \leq \underline{\delta}'$, then the comparison of the developed expressions for the two thresholds implies that $\pi_i^d(a_m) - \pi_i^d(\underline{a}_P) \leq \pi_m - \bar{\pi}$. Since $\bar{\pi} = \pi_m - \pi_i^d(a_m)$ by definition, we have $\pi_i^d(\underline{a}_P) \geq 0$. Since $V_0(\underline{\mathbf{a}}_P, \delta) = \underline{\pi} + \delta V_1(\underline{\mathbf{a}}_P, \delta)$, with $V_1(\underline{\mathbf{a}}_P, \delta)$ as in (20), and $\underline{\mathbf{a}}_P$ as defined above in (1), we have $V_0(\underline{\mathbf{a}}_P, \delta) \geq 0$, which says that the participation constraint (MPC) is also satisfied for $\mathbf{a}_P = \underline{\mathbf{a}}_P$ and $\delta \geq \underline{\delta}'$. This says that a_m is implementable with a finite punishment scheme for all $\delta \geq \underline{\delta}'$.

Then recall from Lemma 7 that the lowest δ compatible with $(MIC\ 1-MIC\ 2)$ and (MSC) is $\underline{\delta}'$. It follows that $\underline{\delta}'$ is the lowest possible discount factor that implements a_m .

($\bar{\delta} > \underline{\delta}'$)

(1) We proceed as in the previous case to define implicitly $\bar{\mathbf{a}}_P$ by

$$V_1(\bar{\mathbf{a}}_P, \delta) = -\frac{\underline{\pi}}{\delta}. \quad (22)$$

Again, we must check that $\bar{\mathbf{a}}_P$ satisfies the sufficient condition introduced in Lemma 8, that is $\underline{\pi} < -\frac{(1-\delta)}{\delta}\underline{\pi} \leq \pi_m$, for all $\delta \geq \bar{\delta}$. The left-hand side inequality is always satisfied for $\delta \in (0, 1]$. On the right-hand side, $\underline{a}_P \succeq \bar{a}_P$ implies that $\underline{\pi} \geq \bar{\pi} = \pi_m - \pi_i^d(a_m)$. As a result $-\frac{(1-\bar{\delta})}{\bar{\delta}}\underline{\pi} \leq \pi_m$, which extends to any $\delta \geq \bar{\delta}$ by monotonicity. Hence there exists at least one $\bar{\mathbf{a}}_P$ for a finite l such that $V_1(\bar{\mathbf{a}}_P, \delta)$ satisfies (22) for any $\delta \geq \bar{\delta}$.

(2) At $\delta = \bar{\delta}$, we check that $(MIC\ 1-MIC\ 2)$ are satisfied for $\mathbf{a}_P = \bar{\mathbf{a}}_P$ (which is as defined in Lemma 8, so that $\bar{a}_{P,1} = \underline{a}_P$, and (MSC) is satisfied by construction). Indeed $X(\bar{\delta}) =$

$V_0(\mathbf{a}_m, \bar{\delta}) - V_0(\bar{\mathbf{a}}_P, \bar{\delta}) < Y(\bar{\delta}, \underline{a}_P)$ with $X(\bar{\delta}) = \pi_i^d(a_m)$, $Y(\bar{\delta}, \underline{a}_P) = \pi_i^d(a_m) \left(1 - \frac{\pi_i^d(\underline{a}_P)}{\pi_m}\right) > \pi_i^d(a_m)$ since $\pi_i^d(\underline{a}_P) < 0$, and $V_0(\mathbf{a}_m, \bar{\delta}) - V_0(\bar{\mathbf{a}}_P, \bar{\delta}) = V_0(\mathbf{a}_m, \bar{\delta}) = \frac{\pi_m}{1-\bar{\delta}} = \pi_i^d(a_m)$. Again, $V_{k+1}(\bar{\mathbf{a}}_P, \delta)$ is strictly increasing in k as long as $k \leq l-1$. Since $\pi_i^d(a_{P,k}) - \pi(a_{P,k})$ is identical for all $1 \leq k \leq l-1$, if (MIC 2) is satisfied, it must be the case all constraints (MIC 3), ..., (MIC l) are also satisfied. There is no loss of generality in assuming that $a_{P,l} \prec a_m$. (If there is equality, collusion can be implemented by the means of a $l-1$ punishment scheme where $a_{P,l-1} = \underline{a}_P \prec a_m$). Assuming this is the case, we know from Assumption (A7) and Lemma 2 that $\pi_i^d(a_{P,l}) - \pi(a_{P,l}) < \pi_i^d(a_m) - \pi_m$. If (MIC 1 holds and is binding, it must be the case that (MIC 1+1) holds also and is slack. We obtain that all incentive constraints are satisfied. Again, since for $\mathbf{a}_P = \bar{\mathbf{a}}_P$ all MICs (MIC 1-MIC 1+1) are monotone increasing in δ , this also holds for all $\delta \geq \bar{\delta}$.

(3) By construction, from (22), $V_0(\bar{\mathbf{a}}_P, \delta) = 0$ hence (MPC) is satisfied for all δ . Given the structure of $\bar{\mathbf{a}}_P$, (MSC) is also satisfied. This says that a_m is implementable with a finite punishment scheme for all $\delta \geq \bar{\delta}$.

Then recall from Lemma 6 that the lowest δ compatible with (MIC 1-MIC 2) and (MPC) is $\bar{\delta}$. It follows that $\bar{\delta}$ is the lowest possible discount factor that implements a_m . ■

7.3 A Linear Example

Inverse demand functions for firm i and all other symmetric firms j are given by (9) and (10). Therefore symmetric profits are

$$\pi(q) = \begin{cases} (1 - q(1 + \gamma(n-1)) - c)q & \text{if } q \leq \underline{q}_P \equiv \frac{1}{(1+\gamma(n-1))} \\ -cq & \text{if } q \geq \frac{1}{1+\gamma(n-1)} \end{cases}, \quad (23)$$

where the piecewise structure results from the non-negativity constraint we impose on prices (solve $1 - q_i - \gamma(n-1)q_j \geq 0$ for $q_i = q_j = q$ to find $q \leq \underline{q}_P \equiv \frac{1}{(1+\gamma(n-1))}$). The collusive quantity and corresponding profits are $q_m = \frac{1-c}{2(1+\gamma(n-1))}$ and $\pi_m = \frac{(1-c)^2}{4(1+\gamma(n-1))}$, respectively.

The one-shot best deviation profits are

$$\pi_i^d(q) = \begin{cases} \frac{1}{4}(1 - c - \gamma(n-1)q)^2 & \text{if } q \leq \tilde{q}_P \equiv \frac{1-c}{\gamma(n-1)} \\ 0 & \text{otherwise} \end{cases}, \quad (24)$$

where \tilde{q}_P is the solution to $\pi_i^d(q) = 0$ (here $f = 0$ implies $\tilde{q}_P = \hat{q}_P$, see (A5) and (A6)). Since $q_m < \tilde{q}_P$ for all parameter values, firm i 's best-reply profits, when each firm in $N \setminus \{i\}$ sells q_m , are $\pi_i^d(q_m) = \frac{(1-c)^2}{16} \frac{(\gamma(n-1)+2)^2}{(1+\gamma(n-1))^2}$, from (24).

For all $q_P > q_m$ one must consider the two forms of $\pi_i^d(q_P)$, that depend on the comparison of q_P with \tilde{q}_P . This leads to two cases:

- (1) If $q_P \leq \tilde{q}_P \equiv \frac{1-c}{\gamma(n-1)}$ best-reply profits are $\pi_i^d(q_P) = \frac{1}{4} (1 - c - \gamma(n-1)) q^2$ and (PC) is slack. When only $(IC1)$ and $(IC2)$ are considered, we know (from Lemma 1) that the optimal punishment q_P^* is a solution in q_P of $\pi_i^d(q_P) - \pi(q_P) = \pi_i^d(q_m) - \pi_m$. There are two solutions. The first one is obviously q_m , which does not apply as a punishment; the second one is

$$q_P^* = \frac{1-c}{2} \frac{3\gamma(n-1)+2}{[2+\gamma(n-1)][1+\gamma(n-1)]}.$$

Here the latter punishment quantity is defined only when lower than \tilde{q}_P , which holds if and only if $\gamma \leq \inf\{\tilde{\gamma}, 1\}$, where $\tilde{\gamma} \equiv 2\frac{1+\sqrt{2}}{n-1}$ and $\inf\{\tilde{\gamma}, 1\} = 1$ if and only if $n < 6$. The threshold value for δ is

$$\delta^* = \frac{1}{16} \frac{[2+\gamma(n-1)]^2}{1+\gamma(n-1)} < 1. \quad (25)$$

This is Regime 1 (see (2)). Next, we find $q_P^* \leq \underline{q}_P$, so that the price $p_i(q_P^*, q_P^*)$ is non-negative and (SC) is slack if and only if $c \geq \underline{c}'$, with

$$\underline{c}' \equiv \frac{\gamma(n-1)-2}{3\gamma(n-1)+2}.$$

The frontier \underline{c}' intersects the line $c = 0$ from below at $\gamma = \hat{\gamma} \equiv \frac{2}{n-1}$. Therefore there exists $\underline{c}' > 0$ if and only if $\frac{2}{n-1} < 1$ (one checks that $\hat{\gamma} < \tilde{\gamma}$ for all $n \geq 2$), or equivalently $n > 3$, otherwise $\underline{c}' = 0$ for all parameter values. Whenever $c < \underline{c}'$ we have $\underline{q}_P < q_P^* \leq \tilde{q}_P$ and (SC) binds. (Here $q_P^* \leq \tilde{q}_P$ is implied by $\gamma \leq \inf\{\tilde{\gamma}, 1\}$.) This is regime 3.

- (2) If $q_P > \tilde{q}_P \equiv \frac{1-c}{\gamma(n-1)}$ best-reply profits are $\pi_i^d(q_P) = 0$ and $(IC2)$ is identically equal to (PC) . (This holds because $f = 0$, otherwise $f > 0$ would imply that $(IC2)$ is strictly weaker than (PC) .) It follows from the previous case (where $q_P \leq \tilde{q}_P \equiv \frac{1-c}{\gamma(n-1)}$) that we need only consider $\gamma \geq \tilde{\gamma}$ and $n \geq 6$ to complete the analysis. There are two solutions in q_P to $-\pi(q_P) = \pi_i^d(q_m) - \pi_m$, the equation that defines \bar{q}_P implicitly. The first one is

strictly less than \tilde{q}_P for all $c < 1$, therefore it is not admissible; the second one is then

$$\bar{q}_P = \frac{1-c}{4} \frac{2[1+\gamma(n-1)] + [2+\gamma(n-1)]\sqrt{1+\gamma(n-1)}}{[1+\gamma(n-1)]^2},$$

which we check is strictly higher than \tilde{q}_P for all parameter values. Then the threshold value for δ now is

$$\bar{\delta} = \left(\frac{\gamma(n-1)}{2+\gamma(n-1)} \right)^2 < 1. \quad (26)$$

This is Regime 2 (see (2)). Next, we find $\bar{q}_P < (=) \underline{q}_P$, so that the price $p_i(\bar{q}_P, \bar{q}_P)$ is non-negative and (SC) is slack if and only if $c > (=) \underline{c}''$, with

$$\underline{c}'' \equiv \frac{\sqrt{1+\gamma(n-1)}[2+\gamma(n-1)] - 2[1+\gamma(n-1)]}{\sqrt{1+\gamma(n-1)}[2+\gamma(n-1)] + 2[1+\gamma(n-1)]}.$$

The frontier \underline{c}'' intersects from below the line $c = 0$ if $\gamma = 0$, and $\underline{c}'' > 0$ otherwise. Therefore $\underline{c}'' > 0$ for all $\gamma \geq \check{\gamma}$. Whenever $c < \underline{c}''$ we have $\underline{q}_P < \bar{q}_P \leq q_P^*$ and (SC) binds. (Here $\bar{q}_P \leq q_P^*$ is implied by $\gamma \geq \check{\gamma}$ and $n \geq 6$.) This is regime 3.

The two preceding paragraphs delineate the parameter subsets in which regime 1 and regime 2 apply, respectively. (In the latter case, since $f = 0$, remark that $(IC2)$ being identical to (PC) implies that regimes 1 and 2 actually coincide for all points (n, γ, c) verifying $n \geq 6$, $\check{\gamma} \leq \gamma \leq 1$, and $\underline{c}'' \leq c < 1$.) All points in the parameter set where regime 3 applies were also identified. In the latter regime, the discount threshold $\underline{\delta}$ solves $\pi_i^d(q_m) - \pi_m = \delta \left(\pi_m - \pi(\underline{q}_P) \right)$. As the specific algebraic form of the latter expression does not depend on parameter values, there is a unique

$$\underline{\delta} = \frac{1}{4} \left(\frac{1-c}{1+c} \right)^2 \frac{(n-1)^2 \gamma^2}{1+\gamma(n-1)}, \quad (27)$$

all n, γ, c .

It remains to compute $\underline{\delta}_M$, the discount threshold when (SC) binds and firms design the optimal l -period punishment scheme. We know (from Proposition 4) that $\underline{\delta}_M = \sup\{\underline{\delta}', \bar{\delta}\}$. Again we know from (24) there are two cases: 1) if $\underline{q}_P < \tilde{q}_P$, or equivalently $c < \frac{1}{1+\gamma(n-1)}$, we have $\pi_i^d(\underline{q}_P) = \frac{1}{4} \left(1 - c - \gamma(n-1) \right) \underline{q}_P^2$, which implies that

$$\underline{\delta}_M = \frac{\gamma(n-1)(1-c)^2}{(1+c)[4(1-c) - \gamma(n-1)(3c-1)]} > \bar{\delta}; \quad (28)$$

and 2) if $\underline{q}_P \geq \tilde{q}_P$, or equivalently $c \geq \frac{1}{1+\gamma(n-1)}$, we have $\pi_i^d(\underline{q}_P) = 0$, hence

$$\underline{\delta}_M = \left(\frac{\gamma(n-1)}{2 + \gamma(n-1)} \right)^2, \quad (29)$$

which is the same expression as $\bar{\delta}$ (regime 2), an illustration of Proposition 5. ■

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